HOMOGENIZATION OF NONSTATIONARY SCHRÖDINGER TYPE EQUATIONS WITH PERIODIC COEFFICIENTS

T. A. SUSLINA

ABSTRACT. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$ we consider selfadjoint strongly elliptic second order differential operators $\mathcal{A}_\varepsilon$ with periodic coefficients depending on $\mathbf{x}/\varepsilon$. We study the behavior of the operator exponential $\exp(-i\mathcal{A}_\varepsilon \tau)$, $\tau \in \mathbb{R}$, for small ε. Approximations for this exponential in the $(H^s \to L_2)$-operator norm with a suitable s are obtained. The results are applied to study the behavior of the solution $\mathbf{u}_\varepsilon$ of the Cauchy problem for the Schrödinger type equation $i\partial_\tau \mathbf{u}_\varepsilon = \mathcal{A}_\varepsilon \mathbf{u}_\varepsilon$.

INTRODUCTION

The paper concerns homogenization for periodic differential operators (DOs). A broad literature is devoted to homogenization problems; first, we mention the books [BeLP], [BaPa], [ZhKO].

0.1. The class of operators. We consider selfadjoint elliptic second order DOs in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ admitting a factorization of the form

$$\mathcal{A} = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x}). \tag{0.1}$$

Here $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$ is the $(m \times n)$-matrix first order DO with constant coefficients. We assume that $m \geqslant n$ and that the symbol $b(\boldsymbol{\xi})$ has maximal rank. It is assumed that the matrix-valued functions $g(\mathbf{x})$ (of size $m \times m$) and $f(\mathbf{x})$ (of size $n \times n$) are periodic with respect to some lattice Γ and such that

$$g(\mathbf{x}) > 0; \quad g, g^{-1} \in L_\infty; \quad f, f^{-1} \in L_\infty.$$

It is convenient to start with a narrower class of operators

$$\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \tag{0.2}$$

corresponding to the case where $f = \mathbf{1}$. Many operators of mathematical physics can be represented in the form (0.1) or (0.2); the simplest example

---

is the acoustics operator $\widehat{\mathcal{A}} = -\operatorname{div} g(\mathbf{x})\nabla = \mathbf{D}^* g(\mathbf{x})\mathbf{D}$. This and other examples are discussed in [BSu1] in detail.

Now we introduce the small parameter $\varepsilon > 0$ and denote $\varphi^\varepsilon(\mathbf{x}) = \varphi(\varepsilon^{-1}\mathbf{x})$ for any Γ-periodic function $\varphi(\mathbf{x})$. Consider the operators

$$\mathcal{A}_\varepsilon = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}), \tag{0.3}$$

$$\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \tag{0.4}$$

whose coefficients oscillate rapidly as $\varepsilon \to 0$.

0.2. Operator error estimates for elliptic and parabolic problems in $\mathbb{R}^d$.
In a series of papers [BSu1–4] by M. Sh. Birman and T. A. Suslina, an operator-theoretic approach to homogenization of elliptic equations in $\mathbb{R}^d$ was suggested and developed. This approach is based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory.

The homogenization problem for elliptic equations in $\mathbb{R}^d$ can be regarded as a problem of asymptotic description of the resolvent of $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0$. For definiteness, let us talk about the simpler operators (0.4). In [BSu1], it was shown that the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ converges to the resolvent $(\widehat{\mathcal{A}}^0 + I)^{-1}$ in the L_2-operator norm, as $\varepsilon \to 0$. Here $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the *effective operator* with the constant *effective matrix* g^0. The formula for the effective matrix is well known in homogenization theory; in the case under consideration it is described below in §8. In [BSu1], it was proved that

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C\varepsilon. \tag{0.5}$$

In [BSu2,3], a more accurate approximation of the resolvent of $\widehat{\mathcal{A}}_\varepsilon$ in the $L_2(\mathbb{R}^d; \mathbb{C}^n)$-operator norm with an error $O(\varepsilon^2)$ was obtained, and in [BSu4] an approximation of the same resolvent in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d; \mathbb{C}^n)$ with an error $O(\varepsilon)$ was found. In these approximations, some correction terms of first order (the *correctors*) were taken into account.

Similarly, the homogenization problem for parabolic equations in $\mathbb{R}^d$ can be regarded as a problem of asymptotic description of the operator exponential $\exp(-\mathcal{A}_\varepsilon \tau)$ for $\tau > 0$ and small ε. The operator-theoretic approach was applied to such problems in [Su1-3], [V], [VSu]. In [Su1,2], it was proved that

$$\|\exp(-\widehat{\mathcal{A}}_\varepsilon \tau) - \exp(-\widehat{\mathcal{A}}^0 \tau)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(\tau + \varepsilon^2)^{-1/2}. \tag{0.6}$$

In [V], a more accurate approximation of the operator $\exp(-\widehat{\mathcal{A}}_\varepsilon \tau)$ in the $L_2(\mathbb{R}^d; \mathbb{C}^n)$-operator norm with an error $O(\varepsilon^2)$ for fixed τ was obtained, and in [Su3] approximation of the same operator in the norm of operators acting from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$ with an error $O(\varepsilon)$ for fixed τ was proved. In these approximations, the first order correctors were taken into account.

Even more accurate approximations of the exponential and the resolvent of $\widehat{\mathcal{A}}_\varepsilon$ with the first and second correctors taken into account were found in [VSu]. In [BSu1–4], [Su1-3], [V], [VSu], similar (but more complicated)

results were obtained also for more general operator (0.3); we will not dwell on this.

Estimates of the form (0.5), (0.6) are called *operator error estimates* in homogenization theory. A different approach to operator error estimates (the so called "modified method of the first appproximation") was suggested by V. V. Zhikov. In [Zh], [ZhPas1], the acoustics operator and the operator of elasticity theory (which have the form (0.4)) were studied; approximations for the resolvents in the $(L_2 \to L_2)$-norm with an error $O(\varepsilon)$ and in the $(L_2 \to H^1)$-norm with an error $O(\varepsilon)$ were obtained. In [ZhPas2], estimate (0.6) was proved for the scalar elliptic operator $-\mathrm{div}\, g^\varepsilon(\mathbf{x})\nabla$.

0.3. Operator error estimates for nonstationary Schrödinger type and hyperbolic type equations.

So, in the case of elliptic and parabolic problems, the spectral approach to homogenization is developed in detail. The situation with homogenization of nonstationary Schrödinger type and hyperbolic equations is different. The paper [BSu5] is devoted to such problems. Again, we dwell on the results for the simpler operator (0.4). In operator terms, the behavior of the operator exponential $\exp(-i\tau\widehat{\mathcal{A}}_\varepsilon)$ and the operator cosine $\cos(\tau\widehat{\mathcal{A}}_\varepsilon^{1/2})$ (where $\tau \in \mathbb{R}$) for small ε is studied. For these operators it is impossible to obtain approximations in the $L_2(\mathbb{R}^d; \mathbb{C}^n)$-operator norm, and we are forced to consider the norm of operators acting from the Sobolev space $H^s(\mathbb{R}^d; \mathbb{C}^n)$ (with appropriate s) to $L_2(\mathbb{R}^d; \mathbb{C}^n)$. In [BSu5], the following estimates were proved:

$$\| \exp(-i\tau\widehat{\mathcal{A}}_\varepsilon) - \exp(-i\tau\widehat{\mathcal{A}}^0)\|_{H^3(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant (C_1 + C_2|\tau|)\varepsilon, \qquad (0.7)$$

$$\| \cos(\tau\widehat{\mathcal{A}}_\varepsilon^{1/2}) - \cos(\tau(\widehat{\mathcal{A}}^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant (\widetilde{C}_1 + \widetilde{C}_2|\tau|)\varepsilon. \qquad (0.8)$$

By interpolation, we can also estimate the operator in (0.7) in the $(H^s \to L_2)$-norm by $O(\varepsilon^{s/3})$ (where $0 \leqslant s \leqslant 3$) and the operator in (0.8) in the $(H^s \to L_2)$-norm by $O(\varepsilon^{s/2})$ (where $0 \leqslant s \leqslant 2$). In [BSu5], approximations for the operator exponential and the operator cosine of the more general operator (0.3) were also obtained. Note that for the operators $\exp(-i\tau\widehat{\mathcal{A}}_\varepsilon)$ and $\cos(\tau\widehat{\mathcal{A}}_\varepsilon^{1/2})$ there are no results concerning more accurate approximations with operator error estimates (and with some correctors taken into account).

The question about the sharpness of the resuts (0.7), (0.8) with respect to the type of operator norm (i. e., the order of the Sobolev space) remained open until now.

Let us explain the method of [BSu5]; we comment on the proof of estimate (0.7). Denote $\mathcal{H}_0 := -\Delta$. Clearly, (0.7) is equivalent to

$$\| \left(\exp(-i\tau\widehat{\mathcal{A}}_\varepsilon) - \exp(-i\tau\widehat{\mathcal{A}}^0)\right) (\mathcal{H}_0 + I)^{-3/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant (C_1 + C_2|\tau|)\varepsilon.$$
$$(0.9)$$

In other words, in order to obtain an estimate in the $(L_2 \to L_2)$-norm, we multiply the operator exponential by a "smoothing factor" $(\mathcal{H}_0 + I)^{-3/2}$.

Next, the scaling transformation shows that (0.9) is equivalent to the estimate

$$\left\| \left(\exp(-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}) - \exp(-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0) \right) \varepsilon^3 (\mathcal{H}_0 + \varepsilon^2 I)^{-3/2} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}$$
$$\leqslant (C_1 + C_2|\tau|)\varepsilon. \quad (0.10)$$

To prove (0.10), using the unitary Gelfand transformation, we expand $\widehat{\mathcal{A}}$ in the direct integral of the operators $\widehat{\mathcal{A}}(\mathbf{k})$ acting in $L_2(\Omega; \mathbb{C}^n)$ (where Ω is the cell of the lattice Γ). Here $\widehat{\mathcal{A}}(\mathbf{k})$ is given by the differential expression $b(\mathbf{D} + \mathbf{k})^* g(\mathbf{x}) b(\mathbf{D} + \mathbf{k})$ with periodic boundary conditions; the spectrum of $\widehat{\mathcal{A}}(\mathbf{k})$ is discrete. The family of operators $\widehat{\mathcal{A}}(\mathbf{k})$ is studied by means of the analytic perturbation theory (with respect to the onedimensional parameter $t = |\mathbf{k}|$). For the operators $\widehat{\mathcal{A}}(\mathbf{k})$ the analog of estimate (0.10) is proved with the constants independent of $\mathbf{k}$. Then the inverse Gelfand transformation leads to (0.10). A good deal of considerations in the study of the family $\widehat{\mathcal{A}}(\mathbf{k})$ is done in the abstract operator-theoretic setting.

0.4. Main results of the paper. In the present paper, we study the behavior of the operator exponential $\exp(-i\tau\mathcal{A}_\varepsilon)$ for small ε, and next we apply the results to study the behavior of the solution of the nonstationary Schrödinger type equation. On the one hand, we confirm the sharpness of estimate (0.7) in the following sense. We find a condition on the operator, under which the estimate

$$\| \exp(-i\tau\widehat{\mathcal{A}}_\varepsilon) - \exp(-i\tau\widehat{\mathcal{A}}^0) \|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C(\tau)\varepsilon$$

is false if $s < 3$. It is easy to formulate this condition in the spectral terms. We consider the operator family $\widehat{\mathcal{A}}(\mathbf{k})$ and put $\mathbf{k} = t\boldsymbol{\theta}$, $t = |\mathbf{k}|$, $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. This family is analytic with respect to the parameter t. For $t = 0$ the point $\lambda_0 = 0$ is an eigenvalue of multiplicity n of the "unperturbed" operator $\widehat{\mathcal{A}}(0)$. Then for small t there exist the real-analytic branches of the eigenvalues and the eigenvectors of $\widehat{\mathcal{A}}(t\boldsymbol{\theta})$. For small t the eigenvalues $\lambda_l(t, \boldsymbol{\theta})$, $l = 1, \dots, n$, admit the convergent power series expansions

$$\lambda_l(t, \boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \mu_l(\boldsymbol{\theta})t^3 + \dots, \quad l = 1, \dots, n, \quad (0.11)$$

where $\gamma_l(\boldsymbol{\theta}) > 0$ and $\mu_l(\boldsymbol{\theta}) \in \mathbb{R}$. The condition is that $\mu_l(\boldsymbol{\theta}_0) \neq 0$ for at least one l and at least one point $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$. Examples of the operators satisfying this condition are provided; in particular, one example is of the form $-\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$, where $g(\mathbf{x})$ is Hermitian matrix with complex entries.

On the other hand, we distinguish conditions on the operator under which it is possible to improve the result and obtain the estimate

$$\| \exp(-i\tau\widehat{\mathcal{A}}_\varepsilon) - \exp(-i\tau\widehat{\mathcal{A}}^0) \|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant (\check{C}_1 + \check{C}_2|\tau|)\varepsilon. \quad (0.12)$$

In the case where $n = 1$, for (0.12) it suffices that the coefficient $\mu(\boldsymbol{\theta}) = \mu_1(\boldsymbol{\theta})$ in (0.11) is identically zero. In particular, this is the case for the operator $-\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$, where $g(\mathbf{x})$ is symmetric matrix with real entries. In the matrix

case (i. e., for $n \geqslant 2$), besides the condition that all the coefficients $\mu_l(\boldsymbol{\theta})$ in (0.11) are equal to zero, we impose one more condition in terms of the coefficients $\gamma_l(\boldsymbol{\theta})$, $l = 1, \ldots, n$. The simplest version of this condition is that the branches $\gamma_l(\boldsymbol{\theta})$ must not intersect: for each pair $j \neq l$ either $\gamma_j(\boldsymbol{\theta})$ and $\gamma_l(\boldsymbol{\theta})$ are separated from each other or they coincide identically for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

It turns out that for more general operator (0.3) it is convenient to study the operator exponential sandwiched between appropriate rapidly oscillating factors. Namely, we study the operator $f^\varepsilon e^{-i\tau \mathcal{A}_\varepsilon}(f^\varepsilon)^{-1}$ and obtain analogs of the results described above for this operator.

Next, we apply the results given in operator terms to study the behavior of the solution $\mathbf{u}_\varepsilon(\mathbf{x}, \tau)$, $\mathbf{x} \in \mathbb{R}^d$, $\tau \in \mathbb{R}$, of the following problem

$$i\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x}, \tau) = (\widehat{\mathcal{A}}_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x}, \tau) + \mathbf{F}(\mathbf{x}, \tau), \quad \mathbf{u}_\varepsilon(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}).$$

A more general problem with the operator $\mathcal{A}_\varepsilon$ is also studied.

We apply the general results to specific equations of mathematical physics. In particular, we consider the nonstationary Schrödinger equation $i\partial_\tau u_\varepsilon = -\mathrm{div} g^\varepsilon(\mathbf{x}) \nabla u_\varepsilon(\mathbf{x}, \tau) + \varepsilon^{-2} V^\varepsilon(\mathbf{x}) u_\varepsilon(\mathbf{x}, \tau)$ with the singular potential $\varepsilon^{-2} V^\varepsilon$, and also the twodimensional Pauli equation with the singular magnetic potential.

Similar results have been obtained by the author jointly with M. A. Dorodnyi [DSu] for homogenization of the hyperbolic equations with rapidly oscillating coefficients.

0.5.   Method. The results are obtained by further development of the operator-theoretic approach. We follow the plan described above in Subsection 0.3. Considerations are based on the abstract operator-theoretic scheme. Let us dwell on this. In the abstract setting, we study the family of operators $A(t) = X(t)^* X(t)$ acting in some Hilbert space $\mathfrak{H}$. Here $X(t) = X_0 + tX_1$. (This family is modelling the operator family $\mathcal{A}(\mathbf{k}) = \mathcal{A}(t\boldsymbol{\theta})$, but the parameter $\boldsymbol{\theta}$ is absent in the abstract setting.) It is assumed that the point $\lambda_0 = 0$ is an eigenvalue of $A(0)$ of finite multiplicity n. Then for $|t| \leqslant t^0$ the perturbed operator $A(t)$ has exactly n eigenvalues (counted with multiplicities) on the interval $[0, \delta]$ (here δ and t^0 are controlled explicitly). These eigenvalues and the corresponding eigenvectors are real-analytic functions of t. The coefficients of the corresponding power series expansions for the eigenvalues and the eigenvectors are called *threshold characteristics* of the operator $A(t)$. We distinguish a finite rank operator S (the so called *spectral germ* of the operator family $A(t)$) which acts in the space $\mathfrak{N} = \mathrm{Ker}\, A(0)$. The spectral germ (see the definition in Subsection 1.2 below) contains the information about the threshold characteristics of principal order. Let $F(t)$ be the spectral projection of the operator $A(t)$ for the interval $[0, \delta]$. We rely on the threshold approximations for the projection $F(t)$ and for the operator $A(t)F(t)$ obtained in [BSu1, Chapter 1] and [BSu2]. Note that in [BSu5] the threshold approximations of principal order from [BSu1] were applied:

$F(t)$ was approximated by the projection P onto the subspace $\mathfrak{N}$, and the operator $A(t)F(t)$ was approximated by $t^2 SP$. It turns out that, in order to obtain more subtle results described above in Subsection 0.4, we need to use more accurate threshold approximations obtained in [BSu2]. Moreover, we need to divide the eigenvalues of $A(t)$ into clusters and find more detailed threshold approximations associated with this division (see §2).

In terms of the spectral germ, it is possible to approximate the operator exponential $\exp(-i\varepsilon^{-2}\tau A(t))$ multiplied by an appropriate "smoothing factor". Application of the abstract results leads to the required estimates for differential operators. However, at this step additional difficulties arise. They concern the improvement of the results in the case where all the coefficients $\mu_l(\boldsymbol{\theta})$ are equal to zero. These difficulties are related to the fact that in the general (matrix) case we are not always able to make our constructions and estimates uniform in $\boldsymbol{\theta}$, and we are forced to impose the additional assumptions of isolation of the branches $\gamma_l(\boldsymbol{\theta})$, $l = 1, \ldots, n$.

0.6. The scalar case. For the case of the scalar elliptic operator $\widehat{\mathcal{A}}_\varepsilon = -\mathrm{div}\, g^\varepsilon(\mathbf{x})\nabla$ in $L_2(\mathbb{R}^d)$ main results of the paper can be obtained by a simpler method based on analyticity of the operator family $\widehat{\mathcal{A}}(\mathbf{k})$ with respect to the multidimensional parameter $\mathbf{k}$ and variational arguments. This is the subject of the joint paper by I. Kamotski and the author [KamSu].

0.7. Plan of the paper. The paper consists of three chapters. Chapter 1 (§1–5) contains the necessary operator-theoretic material. In Chapter 2 (§6–12), the periodic differential operators of the form (0.1), (0.2) are studied. In §6 we describe the class of operators and introduce the Gelfand transformation. §7 is devoted to the direct integral expansion for periodic operators of the form (0.1); the corresponding family of operators $\mathcal{A}(\mathbf{k})$ acting in $L_2(\Omega; \mathbb{C}^n)$ is incorporated in the framework of the abstract scheme. In §8 we describe the effective characteristics for the operator (0.2). In §9, using the abstract results, we obtain approximation of the smoothed operator exponential $\exp(-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k}))$. The operator $\mathcal{A}(\mathbf{k})$ is considered in §10. In §11, with the help of the abstract results we find approximation of the smoothed sandwiched exponential $\exp(-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k}))$. Next, in §12 we return to the operators (0.1), (0.2) acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$; applying the results of §9 and §11, we obtain approximations of the smoothed operator $\exp(-i\varepsilon^{-2}\tau\widehat{\mathcal{A}})$ and of the smoothed sandwiched operator $\exp(-i\varepsilon^{-2}\tau\mathcal{A})$. Chapter 3 (§13–16) is devoted to homogenization problems. In §13, by the scaling transformation, the results of Chapter 2 imply main results of the paper: approximations for the exponential $\exp(-i\tau\widehat{\mathcal{A}}_\varepsilon)$ and for the sandwiched exponential $\exp(-i\tau\mathcal{A}_\varepsilon)$ in the $(H^s \to L_2)$-norm. In §14, the results are applied to study the behavior of the solution of the Cauchy problem for the nonstationary Schrödinger type equation. The last §15 and §16 are devoted to applications of the general results to the particular equations, namely, to the nonstationary Schrödinger equation and the two-dimensional nonstationary Pauli equation.

0.8. Notation. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. The symbols $(\cdot,\cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ stand for the inner product and the norm in $\mathfrak{H}$, respectively; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ stands for the norm of a linear continuous operator from $\mathfrak{H}$ to $\mathfrak{H}_*$. Sometimes we omit the indices. By $I = I_{\mathfrak{H}}$ we denote the identity operator in $\mathfrak{H}$. If $\mathfrak{N}$ is a subspace in $\mathfrak{H}$, then $\mathfrak{N}^\perp := \mathfrak{H} \ominus \mathfrak{N}$. If P is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{N}$, then $P^\perp$ is the orthogonal projection onto $\mathfrak{N}^\perp$. If $A : \mathfrak{H} \to \mathfrak{H}_*$ is a linear operator, then $\mathrm{Dom}\, A$ stands for its domain and $\mathrm{Ker}\, A$ stands for its kernel.

The symbols $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the inner product and the norm in $\mathbb{C}^n$; $\mathbf{1} = \mathbf{1}_n$ is the unit $(n \times n)$-matrix. If a is an $(n \times n)$-matrix, then the symbol $|a|$ denotes the norm of a viewed as a linear operator in $\mathbb{C}^n$. Next, we use the notation $\mathbf{x} = (x_1,\ldots,x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1,\ldots,d$, $\mathbf{D} = -i\nabla = (D_1,\ldots,D_d)$.

The L_p-classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$, $1 \leqslant p \leqslant \infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O}$ of order s and integrability index p are denoted by $W_p^s(\mathcal{O};\mathbb{C}^n)$. For $p = 2$, we denote this space by $H^s(\mathcal{O};\mathbb{C}^n)$. If $n = 1$, we write simply $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, but sometimes we use such abbreviated notation also for the spaces of vector-valued or matrix-valued functions.

By $c, C, \mathcal{C}, \mathfrak{C}$ (possibly, with indices and marks) we denote various constants in estimates.

<div align="center">Chapter 1. Abstract operator-theoretic scheme</div>

<div align="center">§1. Preliminaries</div>

Our approach to homogenization problems is based on the abstract operator-theoretic scheme.

1.1. The operators $X(t)$ and $A(t)$. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. Suppose that $X_0 : \mathfrak{H} \to \mathfrak{H}_*$ is a densely defined and closed operator, and $X_1 : \mathfrak{H} \to \mathfrak{H}_*$ is a bounded operator. Then the operator $X(t) := X_0 + tX_1$, $t \in \mathbb{R}$, is closed on the domain $\mathrm{Dom}\, X(t) = \mathrm{Dom}\, X_0$. In the abstract setting, the family of selfadjoint (and nonnegative) operators

$$A(t) := X(t)^* X(t), \quad t \in \mathbb{R}, \tag{1.1}$$

in $\mathfrak{H}$ is our main object. The operator (1.1) is generated by the closed quadratic form $\|X(t)u\|_{\mathfrak{H}_*}^2$, $u \in \mathrm{Dom}\, X_0$. Denote $A(0) = X_0^* X_0 =: A_0$ and $\mathfrak{N} := \mathrm{Ker}\, A_0 = \mathrm{Ker}\, X_0$. We impose the following condition.

Condition 1.1. *The point* $\lambda_0 = 0$ *is an isolated point of the spectrum of* A_0, *and* $0 < n := \dim \mathfrak{N} < \infty$.

By d^0 we denote the *distance from the point* $\lambda_0 = 0$ *to the rest of the spectrum of* A_0. We put $\mathfrak{N}_* := \mathrm{Ker}\, X_0^*$, $n_* := \dim \mathfrak{N}_*$, and *assume that* $n \leqslant n_* \leqslant \infty$. Let P and P_* be the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{N}$ and of $\mathfrak{H}_*$ onto $\mathfrak{N}_*$, respectively.

The operator family $A(t)$ has been studied in [BSu1, Chapter 1; BSu2; BSu4, Chapter 1] in detail.

Denote by $F(t; [a, b])$ the spectral projection of $A(t)$ for the interval $[a, b]$, and put $\mathfrak{F}(t; [a, b]) := F(t; [a, b])\mathfrak{H}$. *We fix a number $\delta > 0$ such that $8\delta < d^0$.* We often write $F(t)$ in place of $F(t; [0, \delta])$ and $\mathfrak{F}(t)$ in place of $\mathfrak{F}(t; [0, \delta])$. Next, we choose a number $t^0 > 0$ such that

$$t^0 \leqslant \delta^{1/2} \|X_1\|^{-1}. \tag{1.2}$$

According to [BSu1, Chapter 1, (1.3)],

$$F(t; [0, \delta]) = F(t; [0, 3\delta]), \quad \operatorname{rank} F(t; [0, \delta]) = n, \quad |t| \leqslant t^0.$$

1.2. The operators Z, R, and S. Now we introduce some operators appearing in the analytic perturbation theory considerations; see [BSu1, Chapter 1, §1; BSu2, §1].

Let $\omega \in \mathfrak{N}$, and let $\psi = \psi(\omega) \in \operatorname{Dom} X_0 \cap \mathfrak{N}^\perp$ be a (weak) solution of the equation

$$X_0^*(X_0\psi + X_1\omega) = 0.$$

We define a bounded operator $Z : \mathfrak{H} \to \mathfrak{H}$ by the following relation

$$Zu = \psi(Pu), \quad u \in \mathfrak{H}.$$

Note that Z takes $\mathfrak{N}$ to $\mathfrak{N}^\perp$ and $\mathfrak{N}^\perp$ to $\{0\}$.

Let $R : \mathfrak{N} \to \mathfrak{N}_*$ be the operator defined by

$$R\omega = X_0\psi(\omega) + X_1\omega = (X_0Z + X_1)\omega, \quad \omega \in \mathfrak{N}.$$

Another representation for R is given by $R = P_*X_1|_\mathfrak{N}$.

According to [BSu1, Chapter 1, Subsection 1.3], the operator $S := R^*R : \mathfrak{N} \to \mathfrak{N}$ is called the *spectral germ* of the operator family $A(t)$ at $t = 0$. The germ S can be represented as

$$S = PX_1^*P_*X_1|_\mathfrak{N}. \tag{1.3}$$

The spectral germ is said to be *nondegenerate* if $\operatorname{Ker} S = \{0\}$.

We have

$$\|Z\| \leqslant (8\delta)^{-1/2}\|X_1\|, \quad \|R\| \leqslant \|X_1\|, \quad \|S\| \leqslant \|X_1\|^2. \tag{1.4}$$

1.3. The analytic branches of eigenvalues and eigenvectors of $A(t)$. According to the general analytic perturbation theory (see [Ka]), for $|t| \leqslant t^0$ there exist real-analytic functions $\lambda_l(t)$ (the branches of the eigenvalues) and real-analytic $\mathfrak{H}$-valued functions $\varphi_l(t)$ (the branches of the eigenvectors) such that

$$A(t)\varphi_l(t) = \lambda_l(t)\varphi_l(t), \quad l = 1, \ldots, n, \quad |t| \leqslant t^0, \tag{1.5}$$

and the $\varphi_l(t)$, $l = 1, \ldots, n$, form an *orthonormal basis* in $\mathfrak{F}(t)$. Moreover, for $|t| \leqslant t_*$, where $0 < t_* \leqslant t^0$ is *sufficiently small*, we have the following convergent power series expansions:

$$\lambda_l(t) = \gamma_l t^2 + \mu_l t^3 + \ldots, \quad \gamma_l \geqslant 0, \quad \mu_l \in \mathbb{R}, \quad l = 1, \ldots, n, \tag{1.6}$$

$$\varphi_l(t) = \omega_l + t\psi_l^{(1)} + \ldots, \quad l = 1, \ldots, n. \tag{1.7}$$

The elements $\omega_l = \varphi_l(0)$, $l = 1, \ldots, n$, form an orthonormal basis in $\mathfrak{N}$.

Substituting expansions (1.6), (1.7) in (1.5) and comparing the coefficients of powers t and t^2, we arrive at the relations

$$\widetilde{\omega}_l = \psi_l^{(1)} - Z\omega_l \in \mathfrak{N}, \quad l = 1, \ldots, n, \tag{1.8}$$

$$S\omega_l = \gamma_l\omega_l, \quad l = 1, \ldots, n. \tag{1.9}$$

(Cf. [BSu1, Chapter 1, §1]; BSu2, §1].) Thus, the *numbers γ_l and the elements ω_l defined by (1.5)–(1.7) are eigenvalues and eigenvectors of the germ S.* We have

$$P = \sum_{l=1}^{n} (\cdot, \omega_l)\omega_l, \tag{1.10}$$

$$SP = \sum_{l=1}^{n} \gamma_l(\cdot, \omega_l)\omega_l. \tag{1.11}$$

Remark 1.2. Relations (1.9) give another (spectral) definition of the germ. These relations show that the germ of $A(t)$ at $t = 0$ does not depend on the (possibly, nonunique) choice of factorization (1.1). At the same time, invariant representation (1.3) shows that the germ does not depend on the (possibly, nonunique) choice of the (analytic) basis $\{\varphi_l(t)\}$ in $\mathfrak{F}(t)$. If all the eigenvalues γ_l of S are simple, then the "embrios" ω_l in (1.7) are defined by S uniquely (up to phase factors). If there are multiple eigenvalues among γ_l, then the knowledge of S is not sufficient for determining the elements ω_l.

Note that

$$(\widetilde{\omega}_l, \omega_j) + (\omega_l, \widetilde{\omega}_j) = 0, \quad l, j = 1, \ldots, n, \tag{1.12}$$

which follows from the relations $(\varphi_l(t), \varphi_j(t)) = \delta_{lj}$ by substituting (1.7), comparing the coefficients of power t, and taking (1.8) into account.

1.4. Threshold approximations. The spectral projection $F(t)$ and the operator $A(t)F(t)$ are real-analytic operator-valued functions for $|t| \leqslant t^0$. We have

$$F(t) = \sum_{l=1}^{n} (\cdot, \varphi_l(t))\varphi_l(t), \quad |t| \leqslant t^0,$$

$$A(t)F(t) = \sum_{l=1}^{n} \lambda_l(t)(\cdot, \varphi_l(t))\varphi_l(t), \quad |t| \leqslant t^0.$$

Together with (1.6), (1.7), (1.10), and (1.11) this yields the power series expansions $F(t) = P + tF_1 + \ldots$ and $A(t)F(t) = t^2 SP + t^3 K + \ldots$, convergent for $|t| \leqslant t_*$. However, for our purposes not expansions, but approximations (with one or several first terms) with error estimates on the whole interval $|t| \leqslant t^0$ are needed.

The following statement was obtained in [BSu1, Chapter 1, Theorems 4.1 and 4.3]). In what follows, we agree to denote by β_j various absolute constants (which can be controlled explicitly) assuming that $\beta_j \geqslant 1$.

Theorem 1.3. *Suppose that the assumptions of Subsection* 1.1 *are satisfied. Then we have*

$$\|F(t) - P\| \leqslant C_1|t|, \quad |t| \leqslant t^0, \tag{1.13}$$

$$\|A(t)F(t) - t^2 SP\| \leqslant C_2|t|^3, \quad |t| \leqslant t^0. \tag{1.14}$$

Here t^0 is subject to (1.2), *and the constants C_1, C_2 are given by*

$$C_1 = \beta_1 \delta^{-1/2}\|X_1\|, \quad C_2 = \beta_2 \delta^{-1/2}\|X_1\|^3. \tag{1.15}$$

We also need a more precise approximation for the operator $A(t)F(t)$ obtained in [BSu2, Theorem 4.1].

Theorem 1.4. *Suppose that the assumptions of Subsection* 1.1 *are satisfied. Then for $|t| \leqslant t^0$ we have*

$$A(t)F(t) = t^2 SP + t^3 K + \Psi(t), \tag{1.16}$$

and

$$\|\Psi(t)\| \leqslant C_3 t^4, \quad |t| \leqslant t^0. \tag{1.17}$$

The operator K is represented as

$$K = K_0 + N, \quad N = N_0 + N_*,$$

where K_0 takes $\mathfrak{N}$ to $\mathfrak{N}^\perp$ and $\mathfrak{N}^\perp$ to $\mathfrak{N}$, while N_0 and N_* take $\mathfrak{N}$ to itself and $\mathfrak{N}^\perp$ to $\{0\}$. In terms of the power series coefficients, these operators are given by

$$K_0 = \sum_{l=1}^{n} \gamma_l \left((\cdot, Z\omega_l)\omega_l + (\cdot, \omega_l)Z\omega_l \right),$$

$$N_0 = \sum_{l=1}^{n} \mu_l(\cdot, \omega_l)\omega_l, \tag{1.18}$$

$$N_* = \sum_{l=1}^{n} \gamma_l \left((\cdot, \widetilde{\omega}_l)\omega_l + (\cdot, \omega_l)\widetilde{\omega}_l \right). \tag{1.19}$$

In the invariant terms, we have $K_0 = ZSP + SPZ^*$ and $N = Z^* X_1^* RP + (RP)^* X_1 Z$. The constant in (1.17) *is given by $C_3 = \beta_3 \delta^{-1}\|X_1\|^4$. We have*

$$\|K_0\| \leqslant (2\delta)^{-1/2}\|X_1\|^3, \quad \|N\| \leqslant (2\delta)^{-1/2}\|X_1\|^3. \tag{1.20}$$

Note that

$$P^\perp KP = ZSP, \quad PKP = N. \tag{1.21}$$

Remark 1.5. 1°. If $Z = 0$, then $K_0 = 0$, $N = 0$, and $K = 0$.
2°. In the basis $\{\omega_l\}$, the operators N, N_0, and N_* (restricted to $\mathfrak{N}$) are represented by $(n \times n)$-matrices. The operator N_0 is diagonal:

$$(N_0\omega_j, \omega_k) = \mu_j\delta_{jk}, \quad j, k = 1, \ldots, n.$$

The matrix entries of N_* are given by

$$(N_*\omega_j, \omega_k) = \gamma_k(\omega_j, \widetilde{\omega}_k) + \gamma_j(\widetilde{\omega}_j, \omega_k) = (\gamma_j - \gamma_k)(\widetilde{\omega}_j, \omega_k), \quad j, k = 1, \ldots, n.$$

Here we have taken (1.12) into account. It is seen that the diagonal elements of N_* are equal to zero: $(N_*\omega_j, \omega_j) = 0$, $j = 1, \ldots, n$. Moreover,

$$(N_*\omega_j, \omega_k) = 0 \quad \text{if } \gamma_j = \gamma_k.$$

Thus, in the basis $\{\omega_l\}$, the diagonal part of N coincides with N_0, and the off-diagonal part coincides with N_*. Moreover, in the case where some eigenvalues of S are multiple, the off-diagonal elements of N corresponding to different eigenvectors with the same eigenvalue are equal to zero.

$3°$. If $n = 1$, then $N_* = 0$, i. e., $N = N_0$.

1.5. The nondegeneracy condition. Below we impose the following additional condition on the operator $A(t)$.

Condition 1.6. *There exists a constant $c_* > 0$ such that*

$$A(t) \geqslant c_* t^2 I, \quad |t| \leqslant t^0. \tag{1.22}$$

From (1.22) it follows that $\lambda_l(t) \geqslant c_* t^2$, $l = 1, \ldots, n$, for $|t| \leqslant t^0$. By (1.6), this implies

$$\gamma_l \geqslant c_* > 0, \quad l = 1, \ldots, n, \tag{1.23}$$

i. e., the germ S is nondegenerate.

$$\S 2. \text{ The clusters of eigenvalues of } A(t)$$

This section concerns the case where $n \geqslant 2$.

2.1. Renumbering of eigenvalues. Suppose that Condition 1.6 is satisfied. Then the spectrum of the operator SP consists of the eigenvalue $\lambda_0 = 0$ (with the eigenspace $\mathfrak{N}^\perp$) and the eigenvalues $\gamma_1, \ldots, \gamma_n$ satisfying (1.23). Now it is convenient to change the notation tracing the multiplicities of the eigenvalues. Let p be the number of different eigenvalues of the germ. We enumerate these eigenvalues in the increasing order and denote them by γ_j°, $j = 1, \ldots, p$. Their multiplicities are denoted by $k_1, \ldots, k_p$ (obviously, $k_1 + \cdots + k_p = n$). Then, in the previous notation,

$$\gamma_1 = \cdots = \gamma_{k_1} < \gamma_{k_1+1} = \cdots = \gamma_{k_1+k_2} < \cdots < \gamma_{n-k_p+1} = \cdots = \gamma_n.$$

We have $\gamma_1^\circ = \gamma_1 = \cdots = \gamma_{k_1}$, $\gamma_2^\circ = \gamma_{k_1+1} = \cdots = \gamma_{k_1+k_2}$, etc. Let $\mathfrak{N}_j = \mathrm{Ker}\,(S - \gamma_j^\circ I_{\mathfrak{N}})$, $j = 1, \ldots, p$. Then

$$\mathfrak{N} = \sum_{j=1}^{p} \oplus \mathfrak{N}_j.$$

Let P_j be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{N}_j$. Then

$$P = \sum_{j=1}^{p} P_j, \quad P_j P_l = 0 \quad \text{for } j \neq l. \tag{2.1}$$

We also change the notation for the eigenvectors of the germ (which are the "embrios" in (1.7)) dividing them in p parts so that $\omega_1^{(j)}, \ldots, \omega_{k_j}^{(j)}$ correspond

to the eigenvalue γ_j° and form an orthonormal basis in $\mathfrak{N}_j$. (In the previous notation, these are $\omega_{k_1+\cdots+k_{j-1}+1}, \ldots, \omega_{k_1+\cdots+k_j}$.)

We also change the notation for the analytic branches of the eigenvalues and the eigenvectors of $A(t)$. The eigenvalue and the eigenvector whose expansions (1.6) and (1.7) start with the terms $\gamma_j^\circ t^2$ and $\omega_q^{(j)}$ are denoted by $\lambda_q^{(j)}(t)$ and $\varphi_q^{(j)}(t)$, respectively. For $|t| \leqslant t_*$ we have

$$\lambda_q^{(j)}(t) = \gamma_j^\circ t^2 + \mu_q^{(j)} t^3 + \ldots, \quad q = 1, \ldots, k_j,$$

$$\varphi_q^{(j)}(t) = \omega_q^{(j)} + t\psi_q^{(j)} + \ldots, \quad q = 1, \ldots, k_j.$$

2.2. Refinement of threshold approximations. Suppose that Condition 1.6 is satisfied. For $|t| \leqslant t^0$ we consider the following bounded selfadjoint operator in $\mathfrak{H}$:

$$\mathfrak{A}(t) = \begin{cases} t^{-2} A(t) F(t), & t \neq 0, \\ SP, & t = 0. \end{cases}$$

We apply the spectral perturbation theory arguments, treating $\mathfrak{A}(t)$ as a perturbation of the operator SP. By (1.14),

$$\|\mathfrak{A}(t) - SP\| \leqslant C_2 |t|, \quad |t| \leqslant t^0. \tag{2.2}$$

The results about $A(t)$ (see §1) show that for $|t| \leqslant t^0$ the point $\lambda_0 = 0$ is an eigenvalue of the perturbed operator $\mathfrak{A}(t)$ (with the eigenspace $\mathfrak{F}(t)^\perp$), and $\mathfrak{A}(t)$ has positive eigenvalues of total multiplicity n. We divide these positive eigenvalues in p clusters which for small $|t|$ are located near the eigenvalues $\gamma_1^\circ, \ldots, \gamma_p^\circ$ of the unperturbed operator. Clearly, the j-th cluster consists of the eigenvalues $\nu_q^{(j)}(t) = t^{-2} \lambda_q^{(j)}(t)$ of $\mathfrak{A}(t)$, $q = 1, \ldots, k_j$, since $\nu_q^{(j)}(t)$ are continuous (and even analytic) in t and $\nu_q^{(j)}(0) = \gamma_j^\circ$. The corresponding orthonormal eigenvectors are $\varphi_q^{(j)}(t)$, $q = 1, \ldots, k_j$.

For sufficiently small $|t|$ these clusters are separated from each other. However, it will be more convenient for our purposes, for each pair of indices $j \neq l$, to divide the clusters in two parts separated from each other and such that one part contains the j-th cluster and another part contains the l-th cluster. For each pair of indices (j, l), $1 \leqslant j, l \leqslant p$, $j \neq l$, we denote

$$c_{jl}^\circ := \min\{c_*, n^{-1}|\gamma_l^\circ - \gamma_j^\circ|\}. \tag{2.3}$$

Clearly, there exists a number $i_0 = i_0(j, l)$, where $j \leqslant i_0 \leqslant l - 1$ if $j < l$ and $l \leqslant i_0 \leqslant j - 1$ if $l < j$, such that $\gamma_{i_0+1}^\circ - \gamma_{i_0}^\circ \geqslant c_{jl}^\circ$. It means that on the interval between γ_j° and γ_l° there is a gap in the spectrum of S of length at least c_{jl}°. If such i_0 is not unique, we agree to take the minimal possible i_0 (for definiteness).

We choose a number $t_{jl}^{00} \leqslant t^0$ such that (see (1.15))

$$t_{jl}^{00} \leqslant (4C_2)^{-1} c_{jl}^\circ = (4\beta_2)^{-1} \delta^{1/2} \|X_1\|^{-3} c_{jl}^\circ. \tag{2.4}$$

By (2.2), for $|t| \leqslant t_{jl}^{00}$ we have $\|\mathfrak{A}(t) - SP\| \leqslant c_{jl}^{\circ}/4$. Hence, the segments $[\gamma_1^{\circ} - c_{jl}^{\circ}/4, \gamma_{i_0}^{\circ} + c_{jl}^{\circ}/4]$ and $[\gamma_{i_0+1}^{\circ} - c_{jl}^{\circ}/4, \gamma_p^{\circ} + c_{jl}^{\circ}/4]$ are disjoint, and the distance between them is at least $c_{jl}^{\circ}/2$. Consequently, for $|t| \leqslant t_{jl}^{00}$ the perturbed operator $\mathfrak{A}(t)$ has exactly $k_1 + \cdots + k_{i_0}$ eigenvalues (counted with multiplicities) in the segment $[\gamma_1^{\circ} - c_{jl}^{\circ}/4, \gamma_{i_0}^{\circ} + c_{jl}^{\circ}/4]$. These are $\nu_1^{(1)}(t), \ldots, \nu_{k_1}^{(1)}(t); \ldots; \nu_1^{(i_0)}(t), \ldots, \nu_{k_{i_0}}^{(i_0)}(t)$. We denote the corresponding eigenspace by $\mathfrak{F}_{jl}^{(1)}(t)$; for $t \neq 0$ it coincides with the eigenspace $\mathfrak{F}(t; [0, (\gamma_{i_0}^{\circ} + c_{jl}^{\circ}/4)t^2])$ of $A(t)$. The elements $\varphi_1^{(1)}(t), \ldots, \varphi_{k_1}^{(1)}(t); \ldots; \varphi_1^{(i_0)}(t), \ldots, \varphi_{k_{i_0}}^{(i_0)}(t)$ form an orthonormal basis in $\mathfrak{F}_{jl}^{(1)}(t)$. Similarly, for $|t| \leqslant t_{jl}^{00}$ the perturbed operator $\mathfrak{A}(t)$ has exactly $k_{i_0+1} + \cdots + k_p$ eigenvalues (counted with multiplicities) in the segment $[\gamma_{i_0+1}^{\circ} - c_{jl}^{\circ}/4, \gamma_p^{\circ} + c_{jl}^{\circ}/4]$. These are $\nu_1^{(i_0+1)}(t), \ldots, \nu_{k_{i_0+1}}^{(i_0+1)}(t); \ldots; \nu_1^{(p)}(t), \ldots, \nu_{k_p}^{(p)}(t)$. The corresponding eigenspace is denoted by $\mathfrak{F}_{jl}^{(2)}(t)$; for $t \neq 0$ it coincides with the eigenspace $\mathfrak{F}(t; [(\gamma_{i_0+1}^{\circ} - c_{jl}^{\circ}/4)t^2, (\gamma_p^{\circ} + c_{jl}^{\circ}/4)t^2])$ of $A(t)$. The elements $\varphi_1^{(i_0+1)}(t), \ldots, \varphi_{k_{i_0+1}}^{(i_0+1)}(t); \ldots; \varphi_1^{(p)}(t), \ldots, \varphi_{k_p}^{(p)}(t)$ form an orthonormal basis in $\mathfrak{F}_{jl}^{(2)}(t)$. Let $F_{jl}^{(r)}(t)$ be the orthogonal projections onto $\mathfrak{F}_{jl}^{(r)}(t)$, $r = 1, 2$. Then the spectral projection $F(t)$ of the operator $A(t)$ for the interval $[0, \delta]$ can be represented as

$$F(t) = F_{jl}^{(1)}(t) + F_{jl}^{(2)}(t), \quad |t| \leqslant t_{jl}^{00}. \tag{2.5}$$

Proposition 2.1. *For* $|t| \leqslant t_{jl}^{00}$ *we have*

$$\|F_{jl}^{(1)}(t) - (P_1 + \cdots + P_{i_0})\| \leqslant C_{4,jl}|t|, \tag{2.6}$$

$$\|F_{jl}^{(2)}(t) - (P_{i_0+1} + \cdots + P_p)\| \leqslant C_{4,jl}|t|. \tag{2.7}$$

The number t_{jl}^{00} *is subject to* (2.3), (2.4), *and the constant* $C_{4,jl}$ *is given by*

$$C_{4,jl} = \beta_4 \delta^{-1/2} \|X_1\|^5 (c_{jl}^{\circ})^{-2}. \tag{2.8}$$

Proof. Consider the contour $\Gamma_1 \subset \mathbb{C}$ that envelops the interval $[\gamma_1^{\circ} - c_{jl}^{\circ}/4, \gamma_{i_0}^{\circ} + c_{jl}^{\circ}/4]$ equidistantly at the distance $c_{jl}^{\circ}/4$. For $|t| \leqslant t_{jl}^{00}$ this contour encloses the first i_0 clusters of the eigenvalues of $\mathfrak{A}(t)$ and is separated from the other clusters. We have

$$F_{jl}^{(1)}(t) = -\frac{1}{2\pi i} \int_{\Gamma_1} (\mathfrak{A}(t) - zI)^{-1} \, dz, \tag{2.9}$$

$$P_1 + \cdots + P_{i_0} = -\frac{1}{2\pi i} \int_{\Gamma_1} (SP - zI)^{-1} \, dz, \tag{2.10}$$

where we integrate in the positive direction. For $|t| \leqslant t_{jl}^{00}$ and $z \in \Gamma_1$ we have

$$\|(SP - zI)^{-1}\| \leqslant 2(c_{jl}^{\circ})^{-1}, \tag{2.11}$$

$$\|(\mathfrak{A}(t) - zI)^{-1}\| \leqslant 4(c_{jl}^{\circ})^{-1}. \tag{2.12}$$

Next, by the resolvent identity,

$$(\mathfrak{A}(t) - zI)^{-1} - (SP - zI)^{-1} = (\mathfrak{A}(t) - zI)^{-1}(SP - \mathfrak{A}(t))(SP - zI)^{-1}.$$

Combining this with (2.2), (2.11), and (2.12), we obtain

$$\|(\mathfrak{A}(t) - zI)^{-1} - (SP - zI)^{-1}\| \leqslant 8C_2(c_{jl}^{\circ})^{-2}|t|, \quad z \in \Gamma_1, \quad |t| \leqslant t_{jl}^{00}. \tag{2.13}$$

It is easy to estimate the length of the contour Γ_1 by $2\|S\|$. Now, relations (2.9), (2.10), (2.13), (1.4), and (1.15) imply (2.6) with $C_{4,jl} = 8\pi^{-1}C_2\|X_1\|^2(c_{jl}^{\circ})^{-2} = \beta_4 \delta^{-1/2}\|X_1\|^5(c_{jl}^{\circ})^{-2}$.

Estimate (2.7) is proved similarly by integration over the contour Γ_2 that envelops the interval $[\gamma_{i_0+1}^{\circ} - c_{jl}^{\circ}/4, \gamma_p^{\circ} + c_{jl}^{\circ}/4]$ equidistantly at the distance $c_{jl}^{\circ}/4$. $\bullet$

§3. Threshold approximations for the operator exponential

3.1. Approximation of the operator $e^{-i\tau A(t)}P$. In this subsection, we approximate the operator $e^{-i\tau A(t)}P$ by $e^{-i\tau t^2 SP}P$ for $\tau \in \mathbb{R}$ and $|t| \leqslant t^0$. Such approximation was found in [BSu5, §2]. Now we will repeat the proof of this result tracing more carefully how different terms are estimated. This will be needed in what follows to confirm the sharpness of the result. On the other hand, we will distinguish an important case where the result of [BSu5] can be refined.

Consider the operator

$$E(t, \tau) := e^{-i\tau A(t)}P - e^{-i\tau t^2 SP}P. \tag{3.1}$$

We have

$$E(t, \tau) = E_1(t, \tau) + E_2(t, \tau), \tag{3.2}$$

$$E_1(t, \tau) = e^{-i\tau A(t)}F(t)^{\perp}P - F(t)^{\perp}e^{-i\tau t^2 SP}P, \tag{3.3}$$

$$E_2(t, \tau) = e^{-i\tau A(t)}F(t)P - F(t)e^{-i\tau t^2 SP}P. \tag{3.4}$$

Since $F(t)^{\perp}P = (P - F(t))P$, (1.13) implies the following estimate for the operator (3.3):

$$\|E_1(t, \tau)\| \leqslant 2C_1|t|, \quad |t| \leqslant t^0. \tag{3.5}$$

The operator (3.4) can be written as

$$E_2(t, \tau) = e^{-i\tau A(t)}\Sigma(t, \tau), \tag{3.6}$$

$$\Sigma(t, \tau) := F(t)P - e^{i\tau A(t)}F(t)e^{-i\tau t^2 SP}P.$$

Obviously, $\Sigma(t, 0) = 0$ and

$$\Sigma'(t, \tau) := \frac{d\Sigma}{d\tau}(t, \tau) = -ie^{i\tau A(t)}F(t)\left(A(t)F(t) - t^2 SP\right)e^{-i\tau t^2 SP}P. \tag{3.7}$$

Since $\Sigma(t,\tau) = \int_0^\tau \Sigma'(t,\widetilde\tau)\, d\widetilde\tau$, by (3.7) and (1.14), we have

$$\|\Sigma(t,\tau)\| \leqslant C_2|\tau||t|^3, \quad |t| \leqslant t^0. \tag{3.8}$$

Relations (3.1), (3.2), (3.5), (3.6), and (3.8) imply the following result which is close to Theorem 2.1 from [BSu5].

Theorem 3.1. *Under the assumptions of Subsection* 1.1, *for* $\tau \in \mathbb{R}$ *and* $|t| \leqslant t^0$ *we have*

$$\|e^{-i\tau A(t)}P - e^{-i\tau t^2 SP}P\| \leqslant 2C_1|t| + C_2|\tau||t|^3. \tag{3.9}$$

The number t^0 *is subject to* (1.2), *and the constants* C_1 *and* C_2 *are defined by* (1.15).

Now we proceed to more subtle considerations that will allow us to improve the result under the additional assumptions. Using representation (1.16), from (3.7) we obtain

$$\Sigma(t,\tau) = -i \int_0^\tau e^{i\widetilde\tau A(t)} F(t) \left(t^3 K + \Psi(t)\right) e^{-i\widetilde\tau t^2 SP} P \, d\widetilde\tau. \tag{3.10}$$

By (1.21), the operator (3.10) can be represented as

$$\Sigma(t,\tau) = \widetilde\Sigma(t,\tau) + \widehat\Sigma(t,\tau), \tag{3.11}$$

$$\widetilde\Sigma(t,\tau) = -i \int_0^\tau e^{i\widetilde\tau A(t)} F(t) \left(t^3 ZSP + \Psi(t)\right) e^{-i\widetilde\tau t^2 SP} P \, d\widetilde\tau, \tag{3.12}$$

$$\widehat\Sigma(t,\tau) = -it^3 \int_0^\tau e^{i\widetilde\tau A(t)} F(t) N e^{-i\widetilde\tau t^2 SP} P \, d\widetilde\tau.$$

Since $PZ = 0$, then $F(t)ZSP = (F(t) - P)ZSP$. Hence, relations (1.4), (1.13), and (1.17) imply the following estimate for the term (3.12):

$$\|\widetilde\Sigma(t,\tau)\| \leqslant C_5|\tau|t^4, \quad |t| \leqslant t^0, \tag{3.13}$$

where $C_5 = C_1\|X_1\|^3(8\delta)^{-1/2} + C_3 = \beta_5\delta^{-1}\|X_1\|^4$. (We have used expressions for C_1 and C_3.)

Now (3.6) and (3.11) imply that

$$E_2(t,\tau) = \widetilde E_2(t,\tau) + \widehat E_2(t,\tau), \tag{3.14}$$

where $\widetilde E_2(t,\tau) = e^{-i\tau A(t)}\widetilde\Sigma(t,\tau)$ and $\widehat E_2(t,\tau) = e^{-i\tau A(t)}\widehat\Sigma(t,\tau)$. By (3.13),

$$\|\widetilde E_2(t,\tau)\| \leqslant C_5|\tau|t^4, \quad |t| \leqslant t^0, \quad C_5 = \beta_5\delta^{-1}\|X_1\|^4. \tag{3.15}$$

Finally, relations (3.1), (3.2), (3.5), (3.14), (3.15), together with (1.20) imply the following result.

Theorem 3.2. *Under the assumptions of Subsection* 1.1, *for* $\tau \in \mathbb{R}$ *and* $|t| \leqslant t^0$ *we have*

$$e^{-i\tau A(t)}P - e^{-i\tau t^2 SP}P = E_1(t,\tau) + \widetilde E_2(t,\tau) + \widehat E_2(t,\tau),$$

where the first two terms satisfy estimates (3.5) and (3.15), respectively. The third term admits the following representation

$$\widehat{E}_2(t,\tau) = -it^3 e^{-i\tau A(t)} \int_0^\tau e^{i\widetilde{\tau} A(t)} F(t) N e^{-i\widetilde{\tau} t^2 SP} P \, d\widetilde{\tau}, \qquad (3.16)$$

where the operator N is defined in Theorem 1.4. We have

$$\|\widehat{E}_2(t,\tau)\| \leqslant C_6 |\tau||t|^3, \quad |t| \leqslant t^0, \quad C_6 = (2\delta)^{-1/2} \|X_1\|^3. \qquad (3.17)$$

Corollary 3.3. *Suppose that the assumptions of Subsection 1.1 are satisfied. Suppose that $N = 0$. Then for $\tau \in \mathbb{R}$ and $|t| \leqslant t^0$ we have*

$$\|e^{-i\tau A(t)} P - e^{-i\tau t^2 SP} P\| \leqslant 2C_1 |t| + C_5 |\tau| t^4. \qquad (3.18)$$

3.2. Estimate of the term containing N_*. Assume that Condition 1.6 is satisfied. We will use the notation and the results of §2. Recall that $N = N_0 + N_*$. By Remark 1.5, we have

$$P_j N_* P_j = 0, \quad j = 1, \ldots, p; \quad P_l N_0 P_j = 0 \quad \text{for } l \neq j. \qquad (3.19)$$

Thus, relations (2.1) and (3.19) imply the following invariant representations for the operators N_0 and N_*:

$$N_0 = \sum_{j=1}^p P_j N P_j, \quad N_* = \sum_{1 \leqslant l, j \leqslant p: \, j \neq l} P_j N P_l. \qquad (3.20)$$

The term (3.16) can be written as

$$\widehat{E}_2(t,\tau) = E_0(t,\tau) + E_*(t,\tau), \qquad (3.21)$$

where

$$E_0(t,\tau) = -it^3 e^{-i\tau A(t)} \int_0^\tau e^{i\widetilde{\tau} A(t)} F(t) N_0 e^{-i\widetilde{\tau} t^2 SP} P \, d\widetilde{\tau}, \qquad (3.22)$$

$$E_*(t,\tau) = -it^3 e^{-i\tau A(t)} \int_0^\tau e^{i\widetilde{\tau} A(t)} F(t) N_* e^{-i\widetilde{\tau} t^2 SP} P \, d\widetilde{\tau}. \qquad (3.23)$$

In this subsection, we obtain the analog of estimate (3.18) under the weaker assumption that $N_0 = 0$. For this, we have to estimate the operator (3.23). However, we are able to do this only for a smaller interval of t. By (3.20), the term (3.23) can be represented as

$$E_*(t,\tau) = -i e^{-i\tau A(t)} \sum_{1 \leqslant j, l \leqslant p: \, j \neq l} J_{jl}(t,\tau), \qquad (3.24)$$

$$J_{jl}(t,\tau) = t^3 \int_0^\tau e^{i\widetilde{\tau} A(t)} F(t) P_j N P_l e^{-i\widetilde{\tau} t^2 SP} P \, d\widetilde{\tau}. \qquad (3.25)$$

We have to estimate only those terms in (3.24) for which $P_j N P_l \neq 0$. So, let $j \neq l$, and let $P_j N P_l \neq 0$. Suppose that c_{jl}° is defined by (2.3), and t_{jl}^{00} is subject to (2.4). By (2.5), the operator (3.25) can be represented as

$$J_{jl}(t, \tau) = J_{jl}^{(1)}(t, \tau) + J_{jl}^{(2)}(t, \tau), \tag{3.26}$$

$$J_{jl}^{(r)}(t, \tau) = t^3 \int_0^\tau e^{i\widetilde{\tau} A(t)} F_{jl}^{(r)}(t) P_j N P_l e^{-i\widetilde{\tau} t^2 SP} P \, d\widetilde{\tau}, \quad r = 1, 2. \tag{3.27}$$

For definiteness, assume that $j < l$. Then $j < i_0 + 1$ and, by (2.1) and (2.7),

$$\|F_{jl}^{(2)}(t) P_j\| = \| \left(F_{jl}^{(2)}(t) - (P_{i_0+1} + \cdots + P_p) \right) P_j \| \leqslant C_{4,jl} |t|. \tag{3.28}$$

Combining (3.27), (3.28), and (1.20), (2.8), we obtain

$$\|J_{jl}^{(2)}(t, \tau)\| \leqslant C_{4,jl} t^4 |\tau| \|N\| \leqslant C_{7,jl} |\tau| t^4, \quad |t| \leqslant t_{jl}^{00}, \tag{3.29}$$

where $C_{7,jl} = C_{4,jl} (2\delta)^{-1/2} \|X_1\|^3 = \beta_7 \delta^{-1} \|X_1\|^8 (c_{jl}^{\circ})^{-2}$.

It remains to consider the term $J_{jl}^{(1)}(t, \tau)$. Obviously, $P_l e^{-i\widetilde{\tau} t^2 SP} P = e^{-i\widetilde{\tau} \gamma_l^{\circ} t^2} P_l$. Recall that (see Subsection 2.2) the (nonzero) eigenvalues of the operator $A(t) F_{jl}^{(1)}(t)$ are $t^2 \nu_1^{(1)}(t), \ldots, t^2 \nu_{k_1}^{(1)}(t); \ldots; t^2 \nu_1^{(i_0)}(t), \ldots, t^2 \nu_{k_{i_0}}^{(i_0)}(t)$, and $\nu_q^{(r)}(t) \in [\gamma_1^{\circ} - c_{jl}^{\circ}/4, \gamma_{i_0}^{\circ} + c_{jl}^{\circ}/4]$. The corresponding orthonormal eigenvectors are $\varphi_1^{(1)}(t), \ldots, \varphi_{k_1}^{(1)}(t); \ldots; \varphi_1^{(i_0)}(t), \ldots, \varphi_{k_{i_0}}^{(i_0)}(t)$. Then

$$e^{i\widetilde{\tau} A(t)} F_{jl}^{(1)}(t) = \sum_{r=1}^{i_0} \sum_{q=1}^{k_r} e^{i\widetilde{\tau} t^2 \nu_q^{(r)}(t)} (\cdot, \varphi_q^{(r)}(t)) \varphi_q^{(r)}(t).$$

As a result, the operator $J_{jl}^{(1)}(t, \tau)$ can be written as

$$J_{jl}^{(1)}(t, \tau) = t^3 \sum_{r=1}^{i_0} \sum_{q=1}^{k_r} \left(\int_0^\tau e^{i\widetilde{\tau} t^2 (\nu_q^{(r)}(t) - \gamma_l^{\circ})} \, d\widetilde{\tau} \right) (P_j N P_l \cdot, \varphi_q^{(r)}(t)) \varphi_q^{(r)}(t). \tag{3.30}$$

Calculating the integral in (3.30) and taking into account that $|\nu_q^{(r)}(t) - \gamma_l^{\circ}| \geqslant 3 c_{jl}^{\circ}/4$ for $|t| \leqslant t_{jl}^{00}$, we obtain

$$\left| \int_0^\tau e^{i\widetilde{\tau} t^2 (\nu_q^{(r)}(t) - \gamma_l^{\circ})} \, d\widetilde{\tau} \right| \tag{3.31}$$
$$= t^{-2} |\nu_q^{(r)}(t) - \gamma_l^{\circ}|^{-1} \left| e^{i\tau t^2 (\nu_q^{(r)}(t) - \gamma_l^{\circ})} - 1 \right| \leqslant 8 (3 c_{jl}^{\circ})^{-1} t^{-2}.$$

Now relations (3.30) and (3.31) together with (1.20) imply that

$$\|J_{jl}^{(1)}(t, \tau)\| \leqslant 8 (3 c_{jl}^{\circ})^{-1} |t| \|P_j N P_l\| \leqslant C_{8,jl} |t|, \quad |t| \leqslant t_{jl}^{00}, \tag{3.32}$$

where $C_{8,jl} = 8 (2\delta)^{-1/2} \|X_1\|^3 (3 c_{jl}^{\circ})^{-1} = \beta_8 \delta^{-1/2} \|X_1\|^3 (c_{jl}^{\circ})^{-1}$. The case where $j > l$ can be treated similarly.

Let $\mathcal{Z} = \{(j,l) : 1 \leqslant j,l \leqslant p, \ j \neq l, \ P_j N P_l \neq 0\}$. Let c_{jl}° be defined by (2.3). We put

$$c^\circ := \min_{(j,l) \in \mathcal{Z}} c_{jl}^\circ, \tag{3.33}$$

and choose a number $t^{00} \leqslant t^0$ such that

$$t^{00} \leqslant (4\beta_2)^{-1} \delta^{1/2} \|X_1\|^{-3} c^\circ. \tag{3.34}$$

We may assume that $t^{00} \leqslant t_{jl}^{00}$ for all $(j,l) \in \mathcal{Z}$ (see (2.4)).

Now, relations (3.24), (3.26), (3.29), and (3.32), together with expressions for the constants $C_{7,jl}$ and $C_{8,jl}$ yield

$$\|E_*(t,\tau)\| \leqslant C_7 |\tau| t^4 + C_8 |t|, \quad |t| \leqslant t^{00}. \tag{3.35}$$

Here

$$C_7 = \beta_7 n^2 \delta^{-1} \|X_1\|^8 (c^\circ)^{-2}, \quad C_8 = \beta_8 n^2 \delta^{-1/2} \|X_1\|^3 (c^\circ)^{-1}. \tag{3.36}$$

Finally, combining Theorem 3.2 and relations (3.21), (3.22), (3.35), and denoting $\check{E}(t,\tau) := E_1(t,\tau) + \widetilde{E}_2(t,\tau) + E_*(t,\tau)$, we arrive at the following result.

Theorem 3.4. *Suppose that the assumptions of Subsection* 1.1 *and Condition* 1.6 *are satisfied. Suppose that the number* $t^{00} \leqslant t^0$ *is subject to* (3.33), (3.34). *Then for* $\tau \in \mathbb{R}$ *and* $|t| \leqslant t^{00}$ *we have*

$$e^{-i\tau A(t)} P - e^{-i\tau t^2 SP} P = E_0(t,\tau) + \check{E}(t,\tau),$$

where the second term satisfies

$$\|\check{E}(t,\tau)\| \leqslant C_9 |t| + C_{10} |\tau| t^4, \quad |t| \leqslant t^{00}.$$

The constants C_9 *and* C_{10} *are given by* $C_9 = 2C_1 + C_8$, $C_{10} = C_5 + C_7$, *where* C_1, C_5 *are defined by* (1.15), (3.15), *and* C_7, C_8 *are defined by* (3.36). *The operator* $E_0(t,\tau)$ *is given by* (3.22) *and satisfies the estimate*

$$\|E_0(t,\tau)\| \leqslant C_6 |\tau| |t|^3,$$

where C_6 *is as in* (3.17).

Corollary 3.5. *Suppose that the assumptions of Subsection* 1.1 *and Condition* 1.6 *are satisfied. If* $N_0 = 0$, *then for* $\tau \in \mathbb{R}$ *and* $|t| \leqslant t^{00}$ *we have*

$$\|e^{-i\tau A(t)} P - e^{-i\tau t^2 SP} P\| \leqslant C_9 |t| + C_{10} |\tau| t^4.$$

Remark 3.6. Let μ_l, $l = 1, \ldots, n$, be the coefficients at t^3 in the expansions (1.6). By Remark 1.5, the condition $N_0 = 0$ is equivalent to the relations $\mu_l = 0$ for all $l = 1, \ldots, n$.

$$\S 4. \ \text{Approximation of the operator } e^{-i\varepsilon^{-2}\tau A(t)}$$

4.1. Approximation of the operator $e^{-i\varepsilon^{-2}\tau A(t)}$ in the general case.

Let $\varepsilon > 0$. We study the behavior of the operator $e^{-i\varepsilon^{-2}\tau A(t)}$ for small ε. We multiply this operator by the "smoothing factor" $\varepsilon^s(t^2 + \varepsilon^2)^{-s/2}P$, where $s > 0$. (The term is explained by the fact that in applications to differential operators this factor turns into the smoothing operator.) Our goal is to find approximation for the smoothed operator exponential with an error $O(\varepsilon)$ for minimal possible s.

Let $|t| \leqslant t^0$. We apply Theorem 3.1. By (3.9) (with τ replaced by $\varepsilon^{-2}\tau$),

$$\|e^{-i\varepsilon^{-2}\tau A(t)}P - e^{-i\varepsilon^{-2}\tau t^2 SP}P\|\varepsilon^3(t^2 + \varepsilon^2)^{-3/2}$$

$$\leqslant (2C_1|t| + C_2\varepsilon^{-2}|\tau||t|^3)\varepsilon^3(t^2 + \varepsilon^2)^{-3/2} \leqslant (C_1 + C_2|\tau|)\varepsilon.$$

Here we take $s = 3$. We arrive at the following result which has been proved before in [BSu5, Theorem 2.6].

Theorem 4.1. *Suppose that the assumptions of Subsection 1.1 are satisfied. Then for $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leqslant t^0$ we have*

$$\|e^{-i\varepsilon^{-2}\tau A(t)}P - e^{-i\varepsilon^{-2}\tau t^2 SP}P\|\varepsilon^3(t^2 + \varepsilon^2)^{-3/2} \leqslant (C_1 + C_2|\tau|)\varepsilon. \qquad (4.1)$$

The number t^0 is subject to (1.2), and the constants C_1, C_2 are given by (1.15).

4.2. Refinement of approximation for $e^{-i\varepsilon^{-2}\tau A(t)}$ under the additional assumptions.

Corollary 3.3 allows us to improve the result of Theorem 4.1 in the case where $N = 0$.

Theorem 4.2. *Suppose that the assumptions of Theorem 4.1 are satisfied. Suppose that the operator N defined in Theorem 1.4 is equal to zero: $N = 0$. Then for $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leqslant t^0$ we have*

$$\|e^{-i\varepsilon^{-2}\tau A(t)}P - e^{-i\varepsilon^{-2}\tau t^2 SP}P\|\varepsilon^2(t^2 + \varepsilon^2)^{-1} \leqslant (C_1' + C_5|\tau|)\varepsilon. \qquad (4.2)$$

Here $C_1' = \max\{2, C_1\}$ and $C_5 = \beta_5\delta^{-1}\|X_1\|^4$.

Proof. Note that for $|t| \geqslant \sqrt{\varepsilon}$ we have $\varepsilon^2(t^2 + \varepsilon^2)^{-1} \leqslant \varepsilon$, whence the left-hand side of (4.2) does not exceed 2ε.

Thus, we may assume that $|t| < \sqrt{\varepsilon}$. Using (3.18) with τ replaced by $\varepsilon^{-2}\tau$, for $|t| < \sqrt{\varepsilon}$ we obtain

$$\|e^{-i\varepsilon^{-2}\tau A(t)}P - e^{-i\varepsilon^{-2}\tau t^2 SP}P\|\varepsilon^2(t^2 + \varepsilon^2)^{-1}$$

$$\leqslant (2C_1|t| + C_5\varepsilon^{-2}|\tau|t^4)\varepsilon^2(t^2 + \varepsilon^2)^{-1} \leqslant C_1\varepsilon + C_5|\tau|t^2 \leqslant (C_1 + C_5|\tau|)\varepsilon.$$

The required statement follows. $\bullet$

Similarly, Corollary 3.5 yields the following result.

Theorem 4.3. *Suppose that the assumptions of Subsection 1.1 and Condition 1.6 are satisfied. Suppose that the operator N_0 defined in Theorem 1.4 is equal to zero: $N_0 = 0$. Then for $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leqslant t^{00}$ we have*

$$\|e^{-i\varepsilon^{-2}\tau A(t)}P - e^{-i\varepsilon^{-2}\tau t^2 SP}P\|\varepsilon^2(t^2 + \varepsilon^2)^{-1} \leqslant (C_9' + C_{10}|\tau|)\varepsilon.$$

Here the number $t^{00} \leqslant t^0$ is subject to (3.34), the constant C_9' is given by $C_9' = \max\{2, \frac{1}{2}C_9\}$, and C_9, C_{10} are defined in Theorem 3.4.

4.3. Sharpness of the result in the general case. Now we show that the result of Theorem 4.1 is sharp in the general case. Namely, if $N_0 \neq 0$, the exponent s in the smoothing factor can not be taken smaller than 3.

Theorem 4.4. *Let $N_0 \neq 0$. Let $0 \neq \tau \in \mathbb{R}$. Then for any $1 \leqslant s < 3$ it is impossible that the estimate*

$$\|e^{-i\varepsilon^{-2}\tau A(t)}P - e^{-i\varepsilon^{-2}\tau t^2 SP}P\|\varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leqslant C(\tau)\varepsilon \qquad (4.3)$$

holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

Proof. We start with a preliminary remark. Since $F(t)^\perp P = (P - F(t))P$, from (1.13) it follows that

$$\|e^{-i\varepsilon^{-2}\tau A(t)}F(t)^\perp P\|\varepsilon(t^2 + \varepsilon^2)^{-1/2} \leqslant C_1|t|\varepsilon(t^2 + \varepsilon^2)^{-1/2} \leqslant C_1\varepsilon, \quad |t| \leqslant t^0. \tag{4.4}$$

Let us fix $0 \neq \tau \in \mathbb{R}$. We prove by contradiction. Suppose that for some $1 \leqslant s < 3$ there exists a constant $C(\tau) > 0$ such that (4.3) is valid for all sufficiently small $|t|$ and ε. By (4.4), this assumption is equivalent to the existence of a constant $\widetilde{C}(\tau)$ such that

$$\left\| \left(e^{-i\varepsilon^{-2}\tau A(t)}F(t) - e^{-i\varepsilon^{-2}\tau t^2 SP}P \right) P \right\|\varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leqslant \widetilde{C}(\tau)\varepsilon \qquad (4.5)$$

for all sufficiently small $|t|$ and ε.

Consider the interval $|t| \leqslant t_*$ of convergence of the power series expansions (1.6), (1.7). (Now we use the initial enumeration.) We have

$$e^{-i\varepsilon^{-2}\tau A(t)}F(t) = \sum_{l=1}^n e^{-i\varepsilon^{-2}\tau\lambda_l(t)}(\cdot, \varphi_l(t))\varphi_l(t). \qquad (4.6)$$

From the convergence of the power series expansions (1.7) it follows that

$$\|\varphi_l(t) - \omega_l\| \leqslant c_1|t|, \quad |t| \leqslant t_*, \quad l = 1, \ldots, n. \qquad (4.7)$$

Relations (4.6) and (4.7) show that

$$\left\| e^{-i\varepsilon^{-2}\tau A(t)}F(t) - \sum_{l=1}^n e^{-i\varepsilon^{-2}\tau\lambda_l(t)}(\cdot, \omega_l)\omega_l \right\| \leqslant c_2|t|, \quad |t| \leqslant t_*. \qquad (4.8)$$

Comparing (4.8) and (4.5), we see that there exists a constant $\widehat{C}(\tau)$ such that

$$\left\| \sum_{l=1}^n \left(e^{-i\varepsilon^{-2}\tau\lambda_l(t)} - e^{-i\varepsilon^{-2}\tau\gamma_l t^2} \right)(\cdot, \omega_l)\omega_l \right\|\varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leqslant \widehat{C}(\tau)\varepsilon \qquad (4.9)$$

for all sufficiently small $|t|$ and ε.

The condition $N_0 \neq 0$ means that $\mu_j \neq 0$ at least for one j. Applying the operator under the norm sign in (4.9) to ω_j, we obtain

$$\left| e^{-i\varepsilon^{-2}\tau\lambda_j(t)} - e^{-i\varepsilon^{-2}\tau\gamma_j t^2} \right|\varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leqslant \widehat{C}(\tau)\varepsilon \qquad (4.10)$$

for all sufficiently small $|t|$ and ε. The left-hand side of (4.10) can be written as $2|\sin\frac{1}{2}\varepsilon^{-2}\tau(\lambda_j(t)-\gamma_j t^2)|\varepsilon^s(t^2+\varepsilon^2)^{-s/2}$. Using that the expansion (1.6) for $\lambda_j(t)$ is convergent and $\mu_j \neq 0$, we may assume that

$$\frac{1}{2}|\mu_j||t|^3 \leqslant |\lambda_j(t)-\gamma_j t^2| \leqslant \frac{3}{2}|\mu_j||t|^3, \quad |t| \leqslant t_*,$$

possibly diminishing t_*. Hence,

$$\frac{1}{4}\varepsilon^{-2}|\mu_j\tau||t|^3 \leqslant \left|\frac{1}{2}\varepsilon^{-2}\tau(\lambda_j(t)-\gamma_j t^2)\right| \leqslant \frac{3}{4}\varepsilon^{-2}|\mu_j\tau||t|^3, \quad |t| \leqslant t_*.$$

Now, for a fixed $\tau \neq 0$, assuming that ε is sufficiently small (namely, such that $\pi^{1/3}|\mu_j\tau|^{-1/3}\varepsilon^{2/3} \leqslant t_*$), we put $t = t(\varepsilon) = \pi^{1/3}|\mu_j\tau|^{-1/3}\varepsilon^{2/3} = c\,\varepsilon^{2/3}$. For such t we have $2|\sin\frac{1}{2}\varepsilon^{-2}\tau(\lambda_j(t)-\gamma_j t^2)| \geqslant \sqrt{2}$, whence (4.10) implies that $\sqrt{2}\varepsilon^s(c^2\varepsilon^{4/3}+\varepsilon^2)^{-s/2} \leqslant \widehat{C}\varepsilon$. This means that the function $\varepsilon^{s/3-1}(c^2+\varepsilon^{2/3})^{-s/2}$ is uniformly bounded for small ε. But this is not true provided that $s < 3$. This contradiction completes the proof. $\bullet$

§5. Approximation of the sandwiched operator exponential

5.1. The operator family $A(t) = M^*\widehat{A}(t)M$**.** Let $\widehat{\mathfrak{H}}$ be yet another separable Hilbert space. Let $\widehat{X}(t) = \widehat{X}_0 + t\widehat{X}_1 : \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}_*$ be the family of operators of the same form as $X(t)$, and suppose that $\widehat{X}(t)$ satisfies the assumptions of Subsection 1.1. Let $M : \mathfrak{H} \to \widehat{\mathfrak{H}}$ be an isomorphism. Suppose that $M\mathrm{Dom}\,X_0 = \mathrm{Dom}\,\widehat{X}_0$, $X(t) = \widehat{X}(t)M$, and then also $X_0 = \widehat{X}_0 M$, $X_1 = \widehat{X}_1 M$. In $\widehat{\mathfrak{H}}$, we consider the family of operators $\widehat{A}(t) = \widehat{X}(t)^*\widehat{X}(t)$. Then

$$A(t) = M^*\widehat{A}(t)M. \tag{5.1}$$

In what follows, all the objects corresponding to the family $\widehat{A}(t)$ are marked by "hat". Note that $\widehat{\mathfrak{N}} = M\mathfrak{N}$ and $\widehat{\mathfrak{N}}_* = \mathfrak{N}_*$.

In $\widehat{\mathfrak{H}}$ we consider the positive definite operator $Q := (MM^*)^{-1}$. Let $Q_{\widehat{\mathfrak{N}}} = \widehat{P}Q|_{\widehat{\mathfrak{N}}}$ be the block of Q in the subspace $\widehat{\mathfrak{N}}$. Obviously, $Q_{\widehat{\mathfrak{N}}}$ is an isomorphism in $\widehat{\mathfrak{N}}$.

According to [Su2, Proposition 1.2], the orthogonal projection P of $\mathfrak{H}$ onto $\mathfrak{N}$ and the orthogonal projection $\widehat{P}$ of $\widehat{\mathfrak{H}}$ onto $\widehat{\mathfrak{N}}$ satisfy the following relation

$$P = M^{-1}(Q_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}(M^*)^{-1}. \tag{5.2}$$

Let $\widehat{S} : \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}$ be the spectral germ of $\widehat{A}(t)$ at $t = 0$, and let S be the germ of $A(t)$. According to [BSu1, Chapter 1, Subsection 1.5], we have

$$S = PM^*\widehat{S}M|_{\mathfrak{N}}. \tag{5.3}$$

5.2. The operators $\widehat{Z}_Q$ **and** $\widehat{N}_Q$**.** For the operator family $\widehat{A}(t)$ we introduce the operator $\widehat{Z}_Q$ acting in $\widehat{\mathfrak{H}}$ and taking an element $\widehat{u} \in \widehat{\mathfrak{H}}$ to the

solution $\widehat{\psi}_Q$ of the problem

$$\widehat{X}_0^*(\widehat{X}_0\widehat{\psi}_Q + \widehat{X}_1\widehat{\omega}) = 0, \quad Q\widehat{\psi}_Q \perp \widehat{\mathfrak{N}},$$

where $\widehat{\omega} = \widehat{P}\widehat{u}$. As shown in [BSu2, §6], the operator Z for $A(t)$ and the operator $\widehat{Z}_Q$ introduced above satisfy $\widehat{Z}_Q = MZM^{-1}\widehat{P}$. Next, we put

$$\widehat{N}_Q := \widehat{Z}_Q^*\widehat{X}_1^*\widehat{R}\widehat{P} + (\widehat{R}\widehat{P})^*\widehat{X}_1\widehat{Z}_Q. \tag{5.4}$$

According to [BSu2, §6], the operator N for $A(t)$ and the operator (5.4) satisfy

$$\widehat{N}_Q = \widehat{P}(M^*)^{-1}NM^{-1}\widehat{P}. \tag{5.5}$$

Recall that $N = N_0 + N_*$ and introduce the operators

$$\widehat{N}_{0,Q} = \widehat{P}(M^*)^{-1}N_0 M^{-1}\widehat{P}, \quad \widehat{N}_{*,Q} = \widehat{P}(M^*)^{-1}N_* M^{-1}\widehat{P}. \tag{5.6}$$

Then $\widehat{N}_Q = \widehat{N}_{0,Q} + \widehat{N}_{*,Q}$.

Lemma 5.1. *Suppose that the assumptions of Subsection 5.1 are satisfied. Suppose that the operators N and N_0 are defined in Theorem 1.4, and the operators $\widehat{N}_Q$ and $\widehat{N}_{0,Q}$ are defined in Subsection 5.2. Then the relation $N = 0$ is equivalent to the relation $\widehat{N}_Q = 0$. The relation $N_0 = 0$ is equivalent to the relation $\widehat{N}_{0,Q} = 0$.*

Proof. By (5.5), the relation $N = 0$ implies that $\widehat{N}_Q = 0$.

Conversely, let $\widehat{N}_Q = 0$. Then, by (5.5), $\widehat{P}(M^*)^{-1}NM^{-1}\widehat{\omega} = 0$ for any $\widehat{\omega} \in \widehat{\mathfrak{N}}$. Since M^{-1} is an isomorphism of $\widehat{\mathfrak{N}}$ onto $\mathfrak{N}$, then $\widehat{P}(M^*)^{-1}N\omega = 0$ for any $\omega \in \mathfrak{N}$. Multiplying the last relation by $\widehat{\eta} \in \widehat{\mathfrak{N}}$, we obtain $(N\omega, M^{-1}\widehat{\eta}) = 0$ for any $\widehat{\eta} \in \widehat{\mathfrak{N}}$. Using again that $M^{-1}\widehat{\mathfrak{N}} = \mathfrak{N}$, we conclude that the block of N in the subspace $\mathfrak{N}$ is equal to zero. It remains to recall that N takes $\mathfrak{N}$ to $\mathfrak{N}$, and $\mathfrak{N}^\perp$ to $\{0\}$. Hence, $N = 0$.

The second statement can be checked in a similar way. •

5.3. Relations between the operators and the coefficients of the power series expansions. Now we describe relations between the coefficients of the power series expansions (1.6), (1.7) and the operators $\widehat{S}$ and $Q_{\widehat{\mathfrak{N}}}$. (See [BSu3, Subsections 1.6, 1.7].) We denote $\zeta_l := M\omega_l \in \widehat{\mathfrak{N}}$, $l = 1, \ldots, n$. Then relations (1.9), (5.2), and (5.3) show that

$$\widehat{S}\zeta_l = \gamma_l Q_{\widehat{\mathfrak{N}}}\zeta_l, \quad l = 1, \ldots, n. \tag{5.7}$$

The set $\zeta_1, \ldots, \zeta_n$ forms a basis in $\widehat{\mathfrak{N}}$ that is orthonormal with the weight $Q_{\widehat{\mathfrak{N}}}$:

$$(Q_{\widehat{\mathfrak{N}}}\zeta_l, \zeta_j) = \delta_{lj}, \quad l, j = 1, \ldots, n. \tag{5.8}$$

The operators $\widehat{N}_{0,Q}$ and $\widehat{N}_{*,Q}$ can be described in terms of the coefficients of the expansions (1.6) and (1.7); cf. (1.18), (1.19). We put $\widetilde{\zeta}_l := M\widetilde{\omega}_l \in \widehat{\mathfrak{N}}$, $l = 1, \ldots, n$. Then

$$\widehat{N}_{0,Q} = \sum_{k=1}^{n} \mu_k(\cdot, Q_{\widehat{\mathfrak{N}}}\zeta_k)Q_{\widehat{\mathfrak{N}}}\zeta_k, \tag{5.9}$$

$$\widehat{N}_{*,Q} = \sum_{k=1}^{n} \gamma_k \left((\cdot, Q_{\widehat{\mathfrak{N}}}\widetilde{\zeta}_k)Q_{\widehat{\mathfrak{N}}}\zeta_k + (\cdot, Q_{\widehat{\mathfrak{N}}}\zeta_k)Q_{\widehat{\mathfrak{N}}}\widetilde{\zeta}_k \right). \qquad (5.10)$$

Remark 5.2. By (5.8) and (5.9), we have

$$(\widehat{N}_{0,Q}\zeta_j, \zeta_l) = \mu_l\delta_{jl}, \quad j,l = 1,\ldots,n.$$

From (5.8) and (5.10) it follows that

$$(\widehat{N}_{*,Q}\zeta_j, \zeta_l) = \gamma_l(\zeta_j, Q_{\widehat{\mathfrak{N}}}\widetilde{\zeta}_l) + \gamma_j(Q_{\widehat{\mathfrak{N}}}\widetilde{\zeta}_j, \zeta_l), \quad j,l = 1,\ldots,n.$$

Relations (1.12) imply that

$$(Q_{\widehat{\mathfrak{N}}}\widetilde{\zeta}_j, \zeta_l) + (\zeta_j, Q_{\widehat{\mathfrak{N}}}\widetilde{\zeta}_l) = 0, \quad j,l = 1,\ldots,n.$$

Hence,

$$(\widehat{N}_{*,Q}\zeta_j, \zeta_l) = 0 \quad \text{if } \gamma_j = \gamma_l.$$

Now we return to the notation of §2. Recall that the different eigenvalues of the germ S are denoted by γ_j°, $j = 1,\ldots,p$, and the corresponding eigenspaces by $\mathfrak{N}_j$. The vectors $\omega_i^{(j)}$, $i = 1,\ldots,k_j$, form an orthonormal basis in $\mathfrak{N}_j$. Then the same numbers γ_j°, $j = 1,\ldots,p$, are different eigenvalues of the problem (5.7), and $M\mathfrak{N}_j$ are the corresponding eigenspaces. The vectors $\zeta_i^{(j)} = M\omega_i^{(j)}$, $i = 1,\ldots,k_j$, form a basis in $M\mathfrak{N}_j$ (orthonormal with the weight $Q_{\widehat{\mathfrak{N}}}$). By $\mathcal{P}_j$ we denote the "skew" projection onto $M\mathfrak{N}_j$ that is orthogonal with respect to the inner product $(Q_{\widehat{\mathfrak{N}}}\cdot, \cdot)$, i. e.,

$$\mathcal{P}_j = \sum_{i=1}^{k_j}(\cdot, Q_{\widehat{\mathfrak{N}}}\zeta_i^{(j)})\zeta_i^{(j)}, \quad j = 1,\ldots,p.$$

It is easily seen that $\mathcal{P}_j = MP_jM^{-1}\widehat{P}$.

Using (3.20), (5.5), and (5.6), it is easy to check that

$$\widehat{N}_{0,Q} = \sum_{j=1}^{p} \mathcal{P}_j^*\widehat{N}_Q\mathcal{P}_j, \quad \widehat{N}_{*,Q} = \sum_{1\leqslant l,j\leqslant p:\, l\neq j} \mathcal{P}_l^*\widehat{N}_Q\mathcal{P}_j. \qquad (5.11)$$

Relations (5.11) are similar to (3.20); they give the invariant representations for the operators $\widehat{N}_{0,Q}$ and $\widehat{N}_{*,Q}$.

5.4. Approximation of the sandwiched exponential. In this subsection, we find an approximation for the operator exponential $e^{-i\tau A(t)}$ of the family (5.1) in terms of the germ $\widehat{S}$ of $\widehat{A}(t)$ and the isomorphism M. It is convenient to border the exponential by appropriate factors.

We put $M_0 = (Q_{\widehat{\mathfrak{N}}})^{-1/2}$. According to [BSu5, Proposition 3.1], we have

$$Me^{-i\tau t^2 SP}PM^* = M_0e^{-i\tau t^2 M_0\widehat{S}M_0}M_0\widehat{P}. \qquad (5.12)$$

Lemma 5.3. *Under the assumptions of Subsection* 5.1, *we have*

$$\|Me^{-i\tau A(t)}M^{-1}\widehat{P} - M_0e^{-i\tau t^2 M_0\widehat{S}M_0}M_0^{-1}\widehat{P}\|$$
$$\leqslant \|M\|^2\|M^{-1}\|^2\|e^{-i\tau A(t)}P - e^{-i\tau t^2 SP}P\|, \qquad (5.13)$$

$$\|e^{-i\tau A(t)}P - e^{-i\tau t^2 SP}P\|$$
$$\leqslant \|M\|^2\|M^{-1}\|^2\|Me^{-i\tau A(t)}M^{-1}\widehat{P} - M_0 e^{-i\tau t^2 M_0\widehat{S}M_0}M_0^{-1}\widehat{P}\|. \tag{5.14}$$

Proof. Denote the left-hand side of (5.13) by $J(t,\tau)$. Since $M_0 = Q_{\widehat{\mathfrak{N}}}^{-1/2}$, then

$$J(t,\tau) = \|\left(Me^{-i\tau A(t)}M^{-1}Q_{\widehat{\mathfrak{N}}}^{-1}\widehat{P} - M_0 e^{-i\tau t^2 M_0\widehat{S}M_0}M_0\widehat{P}\right)Q_{\widehat{\mathfrak{N}}}\widehat{P}\|.$$

Next, using the identity $M^{-1}Q_{\widehat{\mathfrak{N}}}^{-1}\widehat{P} = PM^*$ (see (5.2)) and (5.12), we obtain

$$J(t,\tau) = \|\left(Me^{-i\tau A(t)}PM^* - Me^{-i\tau t^2 SP}PM^*\right)Q_{\widehat{\mathfrak{N}}}\widehat{P}\|.$$

Hence,

$$J(t,\tau) \leqslant \|M\|^2\|Q_{\widehat{\mathfrak{N}}}\widehat{P}\|\|e^{-i\tau A(t)}P - e^{-i\tau t^2 SP}P\|.$$

Since $\|Q_{\widehat{\mathfrak{N}}}\| \leqslant \|Q\| = \|M^{-1}\|^2$, we arrive at (5.13).

Estimate (5.14) can be checked similarly in the "inverse way". Obviously,

$$\|e^{-i\tau A(t)}P - e^{-i\tau t^2 SP}P\| \leqslant \|M^{-1}\|^2\|Me^{-i\tau A(t)}PM^* - Me^{-i\tau t^2 SP}PM^*\|.$$

By the identity $PM^* = M^{-1}Q_{\widehat{\mathfrak{N}}}^{-1}\widehat{P}$ and (5.12), the right-hand side can be written as $\|M^{-1}\|^2\|Me^{-i\tau A(t)}M^{-1}Q_{\widehat{\mathfrak{N}}}^{-1}\widehat{P} - M_0 e^{-i\tau t^2 M_0\widehat{S}M_0}M_0^{-1}Q_{\widehat{\mathfrak{N}}}^{-1}\widehat{P}\|$. Together with the inequality $\|Q_{\widehat{\mathfrak{N}}}^{-1}\widehat{P}\| \leqslant \|Q^{-1}\| = \|M\|^2$, this implies (5.14). $\bullet$

Now, Theorem 3.1 and inequality (5.13) directly imply the following result (which has been obtained before in [BSu5, Subsection 3.2]).

Theorem 5.4. *Under the assumptions of Subsection* 5.1, *for* $\tau \in \mathbb{R}$ *and* $|t| \leqslant t^0$ *we have*

$$\|Me^{-i\tau A(t)}M^{-1}\widehat{P} - M_0 e^{-i\tau t^2 M_0\widehat{S}M_0}M_0^{-1}\widehat{P}\|$$
$$\leqslant \|M\|^2\|M^{-1}\|^2(2C_1|t| + C_2|\tau||t|^3). \tag{5.15}$$

The number t^0 *is subject to* (1.2), *and the constants* C_1, C_2 *are given by* (1.15).

Similarly, combining Corollary 3.3, Lemma 5.1, and Lemma 5.3, we arrive at the following result.

Theorem 5.5. *Suppose that the assumptions of Subsection* 5.1 *are satisfied. Suppose that the operator* $\widehat{N}_Q$ *defined in Subsection* 5.2 *is equal to zero:* $\widehat{N}_Q = 0$. *Then for* $\tau \in \mathbb{R}$ *and* $|t| \leqslant t^0$ *we have*

$$\|Me^{-i\tau A(t)}M^{-1}\widehat{P} - M_0 e^{-i\tau t^2 M_0\widehat{S}M_0}M_0^{-1}\widehat{P}\|$$
$$\leqslant \|M\|^2\|M^{-1}\|^2(2C_1|t| + C_5|\tau|t^4).$$

The number t^0 *is subject to* (1.2), *and the constants* C_1, C_5 *are defined by* (1.15) *and* (3.15).

Finally, from Corollary 3.5, Lemma 5.1, and Lemma 5.3 we deduce the following statement.

Theorem 5.6. *Suppose that the assumptions of Subsection* 5.1 *and Condition* 1.6 *are satisfied. Suppose that the operator* $\widehat{N}_{0,Q}$ *defined in Subsection* 5.2 *is equal to zero:* $\widehat{N}_{0,Q} = 0$. *Then for* $\tau \in \mathbb{R}$ *and* $|t| \leqslant t^{00}$ *we have*

$$\|Me^{-i\tau A(t)}M^{-1}\widehat{P} - M_0 e^{-i\tau t^2 M_0 \widehat{S}M_0}M_0^{-1}\widehat{P}\| \leqslant \|M\|^2 \|M^{-1}\|^2 (C_9|t| + C_{10}|\tau|t^4).$$

The number t^{00} *is subject to* (3.34), *and the constants* C_9, C_{10} *are as in Theorem* 3.4.

5.5. Approximation of the smoothed sandwiched exponential.

Writing down (5.15) with τ replaced by $\varepsilon^{-2}\tau$ and multiplying it by the "smoothing factor", we arrive at the following result, which has been proved before in [BSu5, Theorem 3.2].

Theorem 5.7. *Under the assumptions of Subsection* 5.1, *for* $\tau \in \mathbb{R}$, $\varepsilon > 0$, *and* $|t| \leqslant t^0$ *we have*

$$\|Me^{-i\varepsilon^{-2}\tau A(t)}M^{-1}\widehat{P} - M_0 e^{-i\varepsilon^{-2}\tau t^2 M_0 \widehat{S}M_0}M_0^{-1}\widehat{P}\|\varepsilon^3(t^2 + \varepsilon^2)^{-3/2}$$
$$\leqslant \|M\|^2 \|M^{-1}\|^2 (C_1 + C_2|\tau|)\varepsilon. \tag{5.16}$$

The number t^0 *is subject to* (1.2), *and the constants* C_1, C_2 *are defined by* (1.15).

Similarly to the proof of Theorem 4.2, from Theorem 5.5 we deduce the following statement.

Theorem 5.8. *Suppose that the assumptions of Subsection* 5.1 *are satisfied. Suppose that the operator* $\widehat{N}_Q$ *defined in Subsection* 5.2 *is equal to zero:* $\widehat{N}_Q = 0$. *Then for* $\tau \in \mathbb{R}$, $\varepsilon > 0$, *and* $|t| \leqslant t^0$ *we have*

$$\|Me^{-i\varepsilon^{-2}\tau A(t)}M^{-1}\widehat{P} - M_0 e^{-i\varepsilon^{-2}\tau t^2 M_0 \widehat{S}M_0}M_0^{-1}\widehat{P}\|\varepsilon^2(t^2 + \varepsilon^2)^{-1}$$
$$\leqslant \|M\|^2 \|M^{-1}\|^2 (C_1' + C_5|\tau|)\varepsilon.$$

The number t^0 *is subject to* (1.2), *and the constants* C_1' *and* C_5 *are as in Theorem* 4.2.

Finally, Theorem 5.6 implies the following result.

Theorem 5.9. *Suppose that the assumptions of Subsection* 5.1 *and Condition* 1.6 *are satisfied. Suppose that the operator* $\widehat{N}_{0,Q}$ *defined in Subsection* 5.2 *is equal to zero:* $\widehat{N}_{0,Q} = 0$. *Then for* $\tau \in \mathbb{R}$, $\varepsilon > 0$, *and* $|t| \leqslant t^{00}$ *we have*

$$\|Me^{-i\varepsilon^{-2}\tau A(t)}M^{-1}\widehat{P} - M_0 e^{-i\varepsilon^{-2}\tau t^2 M_0 \widehat{S}M_0}M_0^{-1}\widehat{P}\|\varepsilon^2(t^2 + \varepsilon^2)^{-1}$$
$$\leqslant \|M\|^2 \|M^{-1}\|^2 (C_9' + C_{10}|\tau|)\varepsilon.$$

The number t^{00} *is subject to* (3.33), (3.34), *and the constants* C_9', C_{10} *are as in Theorem* 4.3.

5.6. The sharpness of the result. Now we confirm that the result of Theorem 5.7 is sharp in the general case. Namely, if $\widehat{N}_{0,Q} \neq 0$, then the exponent s in the smoothing factor can not be taken smaller than 3.

Theorem 5.10. *Suppose that the assumptions of Subsection* 5.1 *are satisfied. Let* $\widehat{N}_{0,Q} \neq 0$. *Let* $0 \neq \tau \in \mathbb{R}$. *Then for any* $1 \leqslant s < 3$ *it is impossible that the estimate*

$$\|Me^{-i\varepsilon^{-2}\tau A(t)}M^{-1}\widehat{P} - M_0 e^{-i\varepsilon^{-2}\tau t^2 M_0 \widehat{S}M_0}M_0^{-1}\widehat{P}\|\varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leqslant C(\tau)\varepsilon \tag{5.17}$$

holds for all sufficiently small $|t|$ *and* $\varepsilon > 0$.

Proof. By Lemma 5.1, under our assumptions we have $N \neq 0$. We prove by contradiction. Let us fix $\tau \neq 0$. Suppose that for some $1 \leqslant s < 3$ there exists a constant $C(\tau) > 0$ such that (5.17) holds for all sufficiently small $|t|$ and ε. By (5.14), this means that the inequality of the form (4.3) also holds (with some other constant). But this contradicts the statement of Theorem 4.4. ●

CHAPTER 2. PERIODIC DIFFERENTIAL OPERATORS IN $L_2(\mathbb{R}^d; \mathbb{C}^n)$

§6. THE CLASS OF OPERATORS.
LATTICES AND THE GELFAND TRANSFORMATION

6.1. Factorized second order operators. Let $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$ be a matrix first order differential operator; here b_l are constant $(m \times n)$-matrices (in general, with complex entries). *Assume that* $m \geqslant n$. Consider the symbol $b(\boldsymbol{\xi}) = \sum_{l=1}^d b_l \xi_l$, $\boldsymbol{\xi} \in \mathbb{R}^d$, *and assume that*

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d. \tag{6.1}$$

Condition (6.1) is equivalent to the inequalities

$$\alpha_0 \mathbf{1}_n \leqslant b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leqslant \alpha_1 \mathbf{1}_n, \quad |\boldsymbol{\theta}| = 1, \quad 0 < \alpha_0 \leqslant \alpha_1 < \infty, \tag{6.2}$$

with some positive constants α_0 and α_1.

Suppose that an $(m \times m)$-matrix-valued function $h(\mathbf{x})$ and an $(n \times n)$-matrix-valued function $f(\mathbf{x})$ (in general, with complex entries) are such that

$$h, h^{-1} \in L_\infty(\mathbb{R}^d); \quad f, f^{-1} \in L_\infty(\mathbb{R}^d). \tag{6.3}$$

Consider the DO

$$\mathcal{X} := hb(\mathbf{D})f : L_2(\mathbb{R}^d; \mathbb{C}^n) \to L_2(\mathbb{R}^d; \mathbb{C}^m),$$

$$\operatorname{Dom}\mathcal{X} := \{\mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^n) : f\mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n)\}.$$

The operator $\mathcal{X}$ is closed. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, consider the selfadjoint operator $\mathcal{A} := \mathcal{X}^*\mathcal{X}$ generated by the closed quadratic form $a[\mathbf{u}, \mathbf{u}] := \|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}$, $\mathbf{u} \in \operatorname{Dom}\mathcal{X}$. Formally, we have

$$\mathcal{A} = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x}), \tag{6.4}$$

where $g(\mathbf{x}) := h(\mathbf{x})^* h(\mathbf{x})$. Note that the Hermitian matrix-valued function $g(\mathbf{x})$ is bounded and uniformly positive definite. Using the Fourier transformation and (6.2), (6.3), it is easy to check that

$$
c' \int\limits_{\mathbb{R}^d} |\mathbf{D}(f\mathbf{u})|^2 \, d\mathbf{x} \leqslant a[\mathbf{u}, \mathbf{u}] \leqslant c'' \int\limits_{\mathbb{R}^d} |\mathbf{D}(f\mathbf{u})|^2 \, d\mathbf{x},
$$
$$
\mathbf{u} \in \operatorname{Dom} \mathcal{X}, \quad c' = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}, \quad c'' = \alpha_1 \|g\|_{L_\infty}. \tag{6.5}
$$

6.2. Lattices in $\mathbb{R}^d$. In what follows, the matrix-valued functions $h(\mathbf{x})$ and $f(\mathbf{x})$ are assumed to be *periodic with respect to some lattice* $\Gamma \subset \mathbb{R}^d$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_d \in \mathbb{R}^d$ be the basis in $\mathbb{R}^d$ that generates the lattice Γ:

$$
\Gamma = \{\mathbf{a} \in \mathbb{R}^d : \ \mathbf{a} = \sum_{j=1}^d \nu^j \mathbf{a}_j, \ \nu^j \in \mathbb{Z}\},
$$

and let Ω be the (elementary) cell of this lattice:

$$
\Omega = \{\mathbf{x} \in \mathbb{R}^d : \ \mathbf{x} = \sum_{j=1}^d \rho^j \mathbf{a}_j, \ 0 < \rho^j < 1\}.
$$

The basis $\mathbf{b}^1, \ldots, \mathbf{b}^d$ in $\mathbb{R}^d$ dual to $\mathbf{a}_1, \ldots, \mathbf{a}_d$ is defined by the relations $\langle \mathbf{b}^i, \mathbf{a}_j \rangle = 2\pi \delta_j^i$. This basis generates the *lattice $\widetilde{\Gamma}$ dual to Γ*:

$$
\widetilde{\Gamma} = \{\mathbf{b} \in \mathbb{R}^d : \ \mathbf{b} = \sum_{i=1}^d \kappa_i \mathbf{b}^i, \ \kappa_i \in \mathbb{Z}\}.
$$

Let $\widetilde{\Omega}$ be the central Brillouin zone of the lattice $\widetilde{\Gamma}$:

$$
\widetilde{\Omega} = \{\mathbf{k} \in \mathbb{R}^d : \ |\mathbf{k}| < |\mathbf{k} - \mathbf{b}|, \ 0 \neq \mathbf{b} \in \widetilde{\Gamma}\}.
$$

Note that $\widetilde{\Omega}$ is a fundamental domain of the lattice $\widetilde{\Gamma}$. Denote $|\Omega| = \operatorname{meas} \Omega$, $|\widetilde{\Omega}| = \operatorname{meas} \widetilde{\Omega}$. We have $|\Omega| \, |\widetilde{\Omega}| = (2\pi)^d$. Let r_0 be the maximal radius of the ball containing in $\operatorname{clos} \widetilde{\Omega}$; then

$$
2r_0 = \min_{0 \neq \mathbf{b} \in \widetilde{\Gamma}} |\mathbf{b}|.
$$

With the lattice Γ, we associate the discrete Fourier transformation $\{\hat{\mathbf{v}}_\mathbf{b}\} \mapsto \mathbf{v}$:

$$
\mathbf{v}(\mathbf{x}) = |\Omega|^{-1/2} \sum_{\mathbf{b} \in \widetilde{\Gamma}} \hat{\mathbf{v}}_\mathbf{b} e^{i\langle \mathbf{b}, \mathbf{x} \rangle}, \quad \mathbf{x} \in \Omega, \tag{6.6}
$$

which is a unitary mapping of $l_2(\widetilde{\Gamma}; \mathbb{C}^n)$ onto $L_2(\Omega; \mathbb{C}^n)$:

$$
\int\limits_{\Omega} |\mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} = \sum_{\mathbf{b} \in \widetilde{\Gamma}} |\hat{\mathbf{v}}_\mathbf{b}|^2.
$$

By $\widetilde{W}_p^s(\Omega)$ we denote the subspace in $W_p^s(\Omega)$ consisting of the functions in $W_p^s(\Omega)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $W_{p,\mathrm{loc}}^s(\mathbb{R}^d)$. If $p = 2$, we use the notation $\widetilde{H}^s(\Omega) = \widetilde{W}_2^s(\Omega)$.

6.3. The Gelfand transformation. Initially, the Gelfand transformation is defined on the functions of the Schwartz class by the formula

$$\widetilde{\mathbf{v}}(\mathbf{k}, \mathbf{x}) = (\mathcal{U}\mathbf{v})(\mathbf{k}, \mathbf{x}) = |\widetilde{\Omega}|^{-1/2} \sum_{\mathbf{a} \in \Gamma} e^{-i\langle \mathbf{k}, \mathbf{x} + \mathbf{a} \rangle} \mathbf{v}(\mathbf{x} + \mathbf{a}),$$

$$\mathbf{v} \in \mathcal{S}(\mathbb{R}^d; \mathbb{C}^n), \quad \mathbf{x} \in \Omega, \quad \mathbf{k} \in \widetilde{\Omega}.$$

Since

$$\int_{\widetilde{\Omega}} \int_{\Omega} |\widetilde{\mathbf{v}}(\mathbf{k}, \mathbf{x})|^2 \, d\mathbf{x} \, d\mathbf{k} = \int_{\mathbb{R}^d} |\mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x},$$

the transformation $\mathcal{U}$ extends by continuity up to a *unitary mapping*

$$\mathcal{U} : L_2(\mathbb{R}^d; \mathbb{C}^n) \to \int_{\widetilde{\Omega}} \oplus L_2(\Omega; \mathbb{C}^n) \, d\mathbf{k} =: \mathcal{K}.$$

The relation $\mathbf{v} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ is equivalent to the fact that $\widetilde{\mathbf{v}}(\mathbf{k}, \cdot) \in \widetilde{H}^1(\Omega; \mathbb{C}^n)$ for almost every $\mathbf{k} \in \widetilde{\Omega}$ and

$$\int_{\widetilde{\Omega}} \int_{\Omega} \left(|(\mathbf{D} + \mathbf{k})\widetilde{\mathbf{v}}(\mathbf{k}, \mathbf{x})|^2 + |\widetilde{\mathbf{v}}(\mathbf{k}, \mathbf{x})|^2 \right) \, d\mathbf{x} \, d\mathbf{k} < \infty.$$

Under the Gelfand transformation $\mathcal{U}$, the operator of multiplication by a bounded periodic function in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ turns into multiplication by the same function on the fibers of the direct integral $\mathcal{K}$. The operator $b(\mathbf{D})$ applied to $\mathbf{v} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ turns into the operator $b(\mathbf{D} + \mathbf{k})$ applied to $\widetilde{\mathbf{v}}(\mathbf{k}, \cdot) \in \widetilde{H}^1(\Omega; \mathbb{C}^n)$.

§7. The direct integral expansion for the operator $\mathcal{A}$

7.1. The forms $a(\mathbf{k})$ and the operators $\mathcal{A}(\mathbf{k})$. Putting $\mathfrak{H} = L_2(\Omega; \mathbb{C}^n)$, $\mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m)$, we consider the closed operator $\mathcal{X}(\mathbf{k}) : \mathfrak{H} \to \mathfrak{H}_*$ depending on the parameter $\mathbf{k} \in \mathbb{R}^d$:

$$\mathcal{X}(\mathbf{k}) = hb(\mathbf{D} + \mathbf{k})f, \quad \mathrm{Dom}\, \mathcal{X}(\mathbf{k}) = \{\mathbf{u} \in \mathfrak{H} : \ f\mathbf{u} \in \widetilde{H}^1(\Omega; \mathbb{C}^n)\} =: \mathcal{D}.$$

The selfadjoint operator $\mathcal{A}(\mathbf{k}) = \mathcal{X}(\mathbf{k})^* \mathcal{X}(\mathbf{k})$ in $\mathfrak{H}$ is generated by the closed quadratic form $a(\mathbf{k})[\mathbf{u}, \mathbf{u}] := \|\mathcal{X}(\mathbf{k})\mathbf{u}\|_{\mathfrak{H}_*}^2$, $\mathbf{u} \in \mathcal{D}$. Using the Fourier series expansion and conditions (6.2), (6.3), it is easy to check that

$$c' \int_{\Omega} |(\mathbf{D} + \mathbf{k})\mathbf{v}|^2 \, d\mathbf{x} \leqslant a(\mathbf{k})[\mathbf{u}, \mathbf{u}] \leqslant c'' \int_{\Omega} |(\mathbf{D} + \mathbf{k})\mathbf{v}|^2 \, d\mathbf{x},$$

$$\mathbf{v} = f\mathbf{u} \in \widetilde{H}^1(\Omega; \mathbb{C}^n),$$

(7.1)

with the same constants c' and c'' as in (6.5). From (7.1) and the compactness of the embedding of $\widetilde{H}^1(\Omega; \mathbb{C}^n)$ in $\mathfrak{H}$ it follows that the resolvent of the operator $\mathcal{A}(\mathbf{k})$ is compact and depends on $\mathbf{k}$ continuously (in the operator norm).

Let

$$\mathfrak{N} := \operatorname{Ker} \mathcal{A}(0) = \operatorname{Ker} \mathcal{X}(0). \tag{7.2}$$

Relations (7.1) with $\mathbf{k} = 0$ show that

$$\mathfrak{N} = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \ f\mathbf{u} = \mathbf{c}, \ \mathbf{c} \in \mathbb{C}^n\}, \quad \dim \mathfrak{N} = n. \tag{7.3}$$

7.2. The band functions. The consecutive eigenvalues $E_j(\mathbf{k})$, $j \in \mathbb{N}$, of the operator $\mathcal{A}(\mathbf{k})$ (counted with multiplicities) are called *band functions*:

$$E_1(\mathbf{k}) \leqslant E_2(\mathbf{k}) \leqslant \ldots \leqslant E_j(\mathbf{k}) \leqslant \ldots, \quad \mathbf{k} \in \mathbb{R}^d.$$

The band functions $E_j(\mathbf{k})$ are continuous and $\widetilde{\Gamma}$-periodic.

We put

$$c_* = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1} \|f^{-1}\|_{L_\infty}^{-2}. \tag{7.4}$$

As shown in [BSu1, Chapter 2, Subsection 2.2] (by simple variational arguments), the band functions satisfy the following estimates:

$$E_j(\mathbf{k}) \geqslant c_* |\mathbf{k}|^2, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}, \quad j = 1, \ldots, n, \tag{7.5}$$

$$E_{n+1}(\mathbf{k}) \geqslant c_* r_0^2, \quad \mathbf{k} \in \operatorname{clos} \widetilde{\Omega}, \tag{7.6}$$

$$E_{n+1}(0) \geqslant 4 c_* r_0^2.$$

7.3. The direct integral expansion for the operator $\mathcal{A}$. With the help of the Gelfand transformation, the operator $\mathcal{A}$ is represented as

$$\mathcal{U} \mathcal{A} \mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{A}(\mathbf{k}) \, d\mathbf{k}. \tag{7.7}$$

This means the following. If $\mathbf{u} \in \operatorname{Dom} a$, then $\widetilde{\mathbf{u}}(\mathbf{k}, \cdot) \in \mathcal{D}$ for almost every $\mathbf{k} \in \widetilde{\Omega}$, and

$$a[\mathbf{u}, \mathbf{u}] = \int_{\widetilde{\Omega}} a(\mathbf{k})[\widetilde{\mathbf{u}}(\mathbf{k}, \cdot), \widetilde{\mathbf{u}}(\mathbf{k}, \cdot)] \, d\mathbf{k}. \tag{7.8}$$

Conversely, if $\widetilde{\mathbf{u}} \in \mathcal{K}$ satisfies $\widetilde{\mathbf{u}}(\mathbf{k}, \cdot) \in \mathcal{D}$ for a. e. $\mathbf{k} \in \widetilde{\Omega}$ and the integral in (7.8) is finite, then $\mathbf{u} \in \operatorname{Dom} a$ and (7.8) is valid.

From (7.7) it follows that the spectrum of $\mathcal{A}$ is the union of segments (spectral bands) $\operatorname{Ran} E_j$, $j \in \mathbb{N}$. By (7.2) and (7.3),

$$\min_{\mathbf{k}} E_j(\mathbf{k}) = E_j(0) = 0, \quad j = 1, \ldots, n,$$

i. e., the first n spectral bands of $\mathcal{A}$ overlap and have the common bottom $\lambda_0 = 0$, while the $(n+1)$-th band is separated from zero (see (7.6)).

7.4. Incorporation of the operators $\mathcal{A}(\mathbf{k})$ into the pattern of §1. For $\mathbf{k} \in \mathbb{R}^d$ we put $\mathbf{k} = t\boldsymbol{\theta}$, $t = |\mathbf{k}|$, $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, and view t as the main parameter.

Then all constructions and estimates will depend on the additional parameter $\boldsymbol{\theta}$, which will often be reflected in the notation. We have to make our estimates uniform in $\boldsymbol{\theta}$.

We will apply the scheme of §1 putting $\mathfrak{H} = L_2(\Omega; \mathbb{C}^n)$ and $\mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m)$. The role of the operator $X(t)$ is played by $X(t, \boldsymbol{\theta}) = \mathcal{X}(t\boldsymbol{\theta})$. Then $X(t, \boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta})$, $\mathrm{Dom}\, X(t, \boldsymbol{\theta}) = \mathrm{Dom}\, X_0 = \mathcal{D}$, where $X_0 = hb(\mathbf{D})f$ and $X_1(\boldsymbol{\theta}) = hb(\boldsymbol{\theta})f$. The role of the operator $A(t)$ is played by $A(t, \boldsymbol{\theta}) = \mathcal{A}(t\boldsymbol{\theta})$. We have $A(t, \boldsymbol{\theta}) = X(t, \boldsymbol{\theta})^* X(t, \boldsymbol{\theta})$. The kernel $\mathfrak{N} = \mathrm{Ker}\, X_0$ is described by (7.3). We have $\dim \mathfrak{N} = n$. Together with (7.6) this shows that Condition 1.1 is satisfied. The distance d^0 from the point $\lambda_0 = 0$ to the rest of the spectrum of $\mathcal{A}(0)$ is equal to $E_{n+1}(0)$ and satisfies the estimate

$$d^0 \geqslant 4c_* r_0^2. \tag{7.9}$$

Here c_* is defined by (7.4). The condition $m \geqslant n$ ensures that $n \leqslant n_*$ (see [BSu1, Chapter 2, §3]).

In Subsection 1.1, it was required to choose the number $\delta \in (0, d^0/8)$. Taking (7.9) into account, we fix δ as follows:

$$\delta = \frac{c_* r_0^2}{4} = \frac{1}{4}\alpha_0 r_0^2 \|g^{-1}\|_{L_\infty}^{-1} \|f^{-1}\|_{L_\infty}^{-2}. \tag{7.10}$$

Next, by (6.2), the operator $X_1(\boldsymbol{\theta}) = hb(\boldsymbol{\theta})f$ satisfies

$$\|X_1(\boldsymbol{\theta})\| \leqslant \alpha_1^{1/2} \|h\|_{L_\infty} \|f\|_{L_\infty}. \tag{7.11}$$

This allows us to take t^0 (see (1.2)) equal to the following number independent of $\boldsymbol{\theta}$:

$$\begin{aligned} t^0 &= \delta^{1/2} \alpha_1^{-1/2} \|h\|_{L_\infty}^{-1} \|f\|_{L_\infty}^{-1} \\ &= \frac{r_0}{2}\alpha_0^{1/2}\alpha_1^{-1/2} \left(\|h\|_{L_\infty} \|h^{-1}\|_{L_\infty} \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}\right)^{-1}. \end{aligned} \tag{7.12}$$

Note that $t^0 \leqslant r_0/2$. Thus, the ball $|\mathbf{k}| \leqslant t^0$ lies inside $\widetilde{\Omega}$. It is important that c_*, δ, and t^0 (see (7.4), (7.10), (7.12)) are independent of $\boldsymbol{\theta}$.

The variational estimates (7.5) for the eigenvalues of $\mathcal{A}(\mathbf{k})$ imply that

$$\mathcal{A}(\mathbf{k}) = A(t, \boldsymbol{\theta}) \geqslant c_* t^2 I, \quad \mathbf{k} = t\boldsymbol{\theta} \in \widetilde{\Omega}.$$

Thus, Condition 1.6 is now satisfied with the constant c_* defined by (7.4). The germ $S(\boldsymbol{\theta})$ of the operator $A(t, \boldsymbol{\theta})$ is nondegenerate uniformly in $\boldsymbol{\theta}$: we have $S(\boldsymbol{\theta}) \geqslant c_* I_{\mathfrak{N}}$ (cf. (1.23)).

§8. THE EFFECTIVE CHARACTERISTICS
OF THE OPERATOR $\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x})b(\mathbf{D})$

8.1. The operator $A(t, \boldsymbol{\theta})$ in the case where $f = \mathbf{1}_n$. In the case where $f = \mathbf{1}_n$, the operator $A(t, \boldsymbol{\theta})$ plays a special role. In this case, all the objects will be marked by "hat". For instance, for the operator

$$\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x})b(\mathbf{D}) \tag{8.1}$$

the family $\widehat{\mathcal{A}}(\mathbf{k})$ is denoted by $\widehat{A}(t, \boldsymbol{\theta})$. The kernel (7.3) takes the form

$$\widehat{\mathfrak{N}} = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n): \ \mathbf{u}(\mathbf{x}) = \mathbf{c}, \ \mathbf{c} \in \mathbb{C}^n\}, \tag{8.2}$$

i. e., $\widehat{\mathfrak{N}}$ consists of constant vector-valued functions. The orthogonal projection $\widehat{P}$ of the space $L_2(\Omega; \mathbb{C}^n)$ onto the subspace (8.2) is the operator of averaging over the cell:

$$\widehat{P}\mathbf{u} = |\Omega|^{-1} \int_{\Omega} \mathbf{u}(\mathbf{x}) \, d\mathbf{x}. \tag{8.3}$$

If $f = \mathbf{1}_n$, the constants (7.4), (7.10), and (7.12) take the form

$$\widehat{c}_* = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1},$$

$$\widehat{\delta} = \frac{r_0^2}{4} \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}, \tag{8.4}$$

$$\widehat{t}^0 = \frac{r_0}{2} \alpha_0^{1/2} \alpha_1^{-1/2} \|g\|_{L_\infty}^{-1/2} \|g^{-1}\|_{L_\infty}^{-1/2}. \tag{8.5}$$

Inequality (7.11) takes the form

$$\|\widehat{X}_1(\boldsymbol{\theta})\| \leqslant \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2}. \tag{8.6}$$

8.2. The germ of the operator $\widehat{A}(t, \boldsymbol{\theta})$. According to [BSu1, Chapter 3, §1], the spectral germ $\widehat{S}(\boldsymbol{\theta})$ of the family $\widehat{A}(t, \boldsymbol{\theta})$ acting in $\widehat{\mathfrak{N}}$ is represented as

$$\widehat{S}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1},$$

where $b(\boldsymbol{\theta})$ is the symbol of the operator $b(\mathbf{D})$, and g^0 is the so called effective matrix. The constant positive $(m \times m)$-matrix g^0 is defined as follows. Suppose that a Γ-periodic $(n \times m)$-matrix-valued function $\Lambda \in \widetilde{H}^1(\Omega)$ is the weak solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_{\Omega} \Lambda(\mathbf{x}) \, d\mathbf{x} = 0. \tag{8.7}$$

We put

$$\widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m). \tag{8.8}$$

Then

$$g^0 = |\Omega|^{-1} \int_{\Omega} \widetilde{g}(\mathbf{x}) \, d\mathbf{x}. \tag{8.9}$$

It turns out that the matrix g^0 is positive definite.

8.3. The effective operator. Consider the symbol

$$\widehat{S}(\mathbf{k}) := t^2 \widehat{S}(\boldsymbol{\theta}) = b(\mathbf{k})^* g^0 b(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^d. \tag{8.10}$$

Expression (8.10) is the symbol of the DO

$$\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) \tag{8.11}$$

acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and called the *effective operator* for the operator $\widehat{\mathcal{A}}$.

Let $\widehat{\mathcal{A}}^0(\mathbf{k})$ be the operator family in $L_2(\Omega; \mathbb{C}^n)$ corresponding to $\widehat{\mathcal{A}}^0$. Then $\widehat{\mathcal{A}}^0(\mathbf{k})$ is given by the expression $b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k})$ with periodic boundary conditions. By (8.3) and (8.10), we have

$$\widehat{S}(\mathbf{k})\widehat{P} = \widehat{\mathcal{A}}^0(\mathbf{k})\widehat{P}. \tag{8.12}$$

8.4. Properties of the effective matrix. The following properties of the matrix g^0 were checked in [BSu1, Chapter 3, Theorem 1.5].

Proposition 8.1. *The effective matrix satisfies the estimates*

$$\underline{g} \leqslant g^0 \leqslant \overline{g}, \tag{8.13}$$

where

$$\overline{g} := |\Omega|^{-1} \int_\Omega g(\mathbf{x}) \, d\mathbf{x}, \quad \underline{g} := \left(|\Omega|^{-1} \int_\Omega g(\mathbf{x}) \, d\mathbf{x} \right)^{-1}.$$

If $m = n$, then $g^0 = \underline{g}$.

For specific DOs, estimates (8.13) are known in homogenization theory as the Voigt-Reuss bracketing. Now we distinguish the cases where one of the inequalities in (8.13) becomes an identity. The following statements were obtained in [BSu1, Chapter 3, Propositions 1.6, 1.7].

Proposition 8.2. *The identity $g^0 = \overline{g}$ is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \ldots, m, \tag{8.14}$$

where $\mathbf{g}_k(\mathbf{x})$, $k = 1, \ldots, m$, are the columns of the matrix $g(\mathbf{x})$.

Proposition 8.3. *The identity $g^0 = \underline{g}$ is equivalent to the representations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k(\mathbf{x}), \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{w}_k \in \widetilde{H}^1(\Omega; \mathbb{C}^n), \quad k = 1, \ldots, m, \tag{8.15}$$

where $\mathbf{l}_k(\mathbf{x})$, $k = 1, \ldots, m$, are the columns of the matrix $g(\mathbf{x})^{-1}$.

8.5. The analytic branches of eigenvalues and eigenvectors. The analytic (in t) branches of the eigenvalues $\widehat{\lambda}_l(t, \boldsymbol{\theta})$ and the analytic branches of the eigenvectors $\widehat{\varphi}_l(t, \boldsymbol{\theta})$ of $\widehat{A}(t, \boldsymbol{\theta})$ admit the power series expansions of the form (1.6) and (1.7) with the coefficients depending on $\boldsymbol{\theta}$:

$$\widehat{\lambda}_l(t, \boldsymbol{\theta}) = \widehat{\gamma}_l(\boldsymbol{\theta})t^2 + \widehat{\mu}_l(\boldsymbol{\theta})t^3 + \ldots, \quad l = 1, \ldots, n, \tag{8.16}$$

$$\widehat{\varphi}_l(t, \boldsymbol{\theta}) = \widehat{\omega}_l(\boldsymbol{\theta}) + t\widehat{\psi}_l^{(1)}(\boldsymbol{\theta}) + \ldots, \quad l = 1, \ldots, n. \tag{8.17}$$

(However, we do not control the interval of convergence $t = |\mathbf{k}| \leqslant t_*(\boldsymbol{\theta})$.) According to (1.9), the numbers $\widehat{\gamma}_l(\boldsymbol{\theta})$ and the elements $\widehat{\omega}_l(\boldsymbol{\theta})$ are eigenvalues and eigenvectors of the germ:

$$b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta})\widehat{\omega}_l(\boldsymbol{\theta}) = \widehat{\gamma}_l(\boldsymbol{\theta})\widehat{\omega}_l(\boldsymbol{\theta}), \quad l = 1, \ldots, n.$$

8.6. The operator $\widehat{N}(\boldsymbol{\theta})$. We need to describe the operator N (that in abstract terms is defined in Theorem 1.4). According to [BSu3,§4], for the family $\widehat{A}(t, \boldsymbol{\theta})$ this operator takes the form

$$\widehat{N}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* L(\boldsymbol{\theta})b(\boldsymbol{\theta})\widehat{P}, \tag{8.18}$$

where the $(m \times m)$-matrix $L(\boldsymbol{\theta})$ is given by

$$L(\boldsymbol{\theta}) = |\Omega|^{-1} \int_{\Omega} (\Lambda(\mathbf{x})^* b(\boldsymbol{\theta})^* \widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^* b(\boldsymbol{\theta}) \Lambda(\mathbf{x})) \, d\mathbf{x}. \qquad (8.19)$$

Here $\Lambda(\mathbf{x})$ is the Γ-periodic solution of problem (8.7), and $\widetilde{g}(\mathbf{x})$ is given by (8.8).

Observe that $L(\mathbf{k}) := tL(\boldsymbol{\theta})$, $\mathbf{k} \in \mathbb{R}^d$, is a Hermitian matrix-valued function first order homogeneous in $\mathbf{k}$. We put $\widehat{N}(\mathbf{k}) := t^3 \widehat{N}(\boldsymbol{\theta})$, $\mathbf{k} \in \mathbb{R}^d$. Then $\widehat{N}(\mathbf{k}) = b(\mathbf{k})^* L(\mathbf{k}) b(\mathbf{k}) \widehat{P}$. The matrix-valued function $b(\mathbf{k})^* L(\mathbf{k}) b(\mathbf{k})$ is a homogeneous third order polynomial of $\mathbf{k} \in \mathbb{R}^d$. Therefore, either $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, or $\widehat{N}(\boldsymbol{\theta}) \neq 0$ at most points $\boldsymbol{\theta}$ (except for the zeroes of this polynomial).

Some cases where the operator (8.18) is equal to zero were distinguished in [BSu3, §4].

Proposition 8.4. *Suppose that at least one of the following conditions is fulfilled*:
1°. *The operator $\widehat{\mathcal{A}}$ has the form $\widehat{\mathcal{A}} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where $g(\mathbf{x})$ is a symmetric matrix with real entries.*
2°. *Relations (8.14) are satisfied, i. e. $g^0 = \overline{g}$.*
3°. *Relations (8.15) are satisfied, i. e. $g^0 = \underline{g}$. (In particular, this is true if $m = n$.)*
Then $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

On the other hand, there are examples (see [BSu3, Subsections 10.4, 13.2, 14.6]) showing that, in general, the operator $\widehat{N}(\boldsymbol{\theta})$ is not equal to zero for the scalar elliptic operator $\mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where $g(\mathbf{x})$ is a Hermitian matrix with complex entries, as well as for matrix operators even with real-valued coefficients; see also Example 8.7 below.

Recall (see Remark 1.5) that $\widehat{N}(\boldsymbol{\theta}) = \widehat{N}_0(\boldsymbol{\theta}) + \widehat{N}_*(\boldsymbol{\theta})$, where the operator $\widehat{N}_0(\boldsymbol{\theta})$ is diagonal in the basis $\widehat{\omega}_1(\boldsymbol{\theta}), \ldots, \widehat{\omega}_n(\boldsymbol{\theta})$ (see (8.17)), while the diagonal elements of the operator $\widehat{N}_*(\boldsymbol{\theta})$ are equal to zero. We have

$$(\widehat{N}(\boldsymbol{\theta}) \widehat{\omega}_l(\boldsymbol{\theta}), \widehat{\omega}_l(\boldsymbol{\theta}))_{L_2(\Omega)} = (\widehat{N}_0(\boldsymbol{\theta}) \widehat{\omega}_l(\boldsymbol{\theta}), \widehat{\omega}_l(\boldsymbol{\theta}))_{L_2(\Omega)} = \widehat{\mu}_l(\boldsymbol{\theta}), \quad l = 1, \ldots, n.$$
$$(8.20)$$

In [BSu3, Subsection 4.3], the following argument is given. Suppose that $b(\boldsymbol{\theta})$ and $g(\mathbf{x})$ are matrices with *real entries*. Then the matrix $\Lambda(\mathbf{x})$ (see (8.7)) has purely imaginary entries, while $\widetilde{g}(\mathbf{x})$ and g^0 are matrices with real entries. In this case $L(\boldsymbol{\theta})$ (see (8.19)) and $b(\boldsymbol{\theta})^* L(\boldsymbol{\theta}) b(\boldsymbol{\theta})$ are Hermitian matrices with purely imaginary entries. Hence, for any *real* vector $\mathbf{q} \in \widehat{\mathfrak{N}}$ we have $(\widehat{N}(\boldsymbol{\theta}) \mathbf{q}, \mathbf{q}) = 0$. If the analytic branches of the eigenvalues $\widehat{\lambda}_l(t, \boldsymbol{\theta})$ and the analytic branches of the eigenvectors $\widehat{\varphi}_l(t, \boldsymbol{\theta})$ of $\widehat{A}(t, \boldsymbol{\theta})$ can be chosen so that the vectors $\widehat{\omega}_1(\boldsymbol{\theta}), \ldots, \widehat{\omega}_n(\boldsymbol{\theta})$ are real, then, by (8.20), we have $\widehat{\mu}_l(\boldsymbol{\theta}) = 0$, $l = 1, \ldots, n$, i. e., $\widehat{N}_0(\boldsymbol{\theta}) = 0$. We arrive at the following statement.

Proposition 8.5. *Suppose that $b(\boldsymbol{\theta})$ and $g(\mathbf{x})$ have real entries. Suppose that in the expansions (8.17) for the analytic branches of the eigenvectors of $\widehat{A}(t, \boldsymbol{\theta})$ the "embrios" $\widehat{\omega}_l(\boldsymbol{\theta})$, $l = 1, \ldots, n$, can be chosen to be real. Then in (8.16) we have $\widehat{\mu}_l(\boldsymbol{\theta}) = 0$, $l = 1, \ldots, n$, i. e., $\widehat{N}_0(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*

In the "real" case under consideration, the germ $\widehat{S}(\boldsymbol{\theta})$ is a symmetric matrix with real entries. Clearly, if the eigenvalue $\widehat{\gamma}_j(\boldsymbol{\theta})$ of the germ is simple, then the embrio $\widehat{\omega}_j(\boldsymbol{\theta})$ is defined uniquely up to a phase factor, and we can always choose $\widehat{\omega}_j(\boldsymbol{\theta})$ to be real. We arrive at the following corollary.

Corollary 8.6. *Suppose that $b(\boldsymbol{\theta})$ and $g(\mathbf{x})$ have real entries. Suppose that the spectrum of the germ $\widehat{S}(\boldsymbol{\theta})$ is simple. Then $\widehat{N}_0(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*

However, as is seen from Example 8.7 considered below, even in the "real" case it is not always possible to choose the vectors $\widehat{\omega}_l(\boldsymbol{\theta})$ to be real. Moreover, it can happen that $\widehat{N}_0(\boldsymbol{\theta}) \neq 0$ at some isolated points $\boldsymbol{\theta}$.

8.7. Multiplicities of the eigenvalues of the germ. Example. Considerations of this subsection concern the case where $n \geqslant 2$. Now we return to the notation of §2, tracing the multiplicities of the eigenvalues of the spectral germ $\widehat{S}(\boldsymbol{\theta})$. In general, the number $p(\boldsymbol{\theta})$ of different eigenvalues $\widehat{\gamma}_1^{\circ}(\boldsymbol{\theta}), \ldots \widehat{\gamma}_{p(\boldsymbol{\theta})}^{\circ}(\boldsymbol{\theta})$ of the spectral germ $\widehat{S}(\boldsymbol{\theta})$ and their multiplicities $k_1(\boldsymbol{\theta}), \ldots, k_{p(\boldsymbol{\theta})}(\boldsymbol{\theta})$ depend on the parameter $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. For a fixed $\boldsymbol{\theta}$ denote by $\widehat{P}_j(\boldsymbol{\theta})$ the orthogonal projection of $L_2(\Omega; \mathbb{C}^n)$ onto the eigenspace of the germ $\widehat{S}(\boldsymbol{\theta})$ corresponding to the eigenvalue $\widehat{\gamma}_j^{\circ}(\boldsymbol{\theta})$. According to (3.20), the operators $\widehat{N}_0(\boldsymbol{\theta})$ and $\widehat{N}_*(\boldsymbol{\theta})$ admit the following invariant representations:

$$\widehat{N}_0(\boldsymbol{\theta}) = \sum_{j=1}^{p(\boldsymbol{\theta})} \widehat{P}_j(\boldsymbol{\theta}) \widehat{N}(\boldsymbol{\theta}) \widehat{P}_j(\boldsymbol{\theta}), \tag{8.21}$$

$$\widehat{N}_*(\boldsymbol{\theta}) = \sum_{1 \leqslant j, l \leqslant p(\boldsymbol{\theta}): \, j \neq l} \widehat{P}_j(\boldsymbol{\theta}) \widehat{N}(\boldsymbol{\theta}) \widehat{P}_l(\boldsymbol{\theta}). \tag{8.22}$$

In conclusion of this section, we consider the example which shows that for matrix operators even with real-valued coefficients the eigenvalues of the germ may be multiple, and the coefficients $\widehat{\mu}_l(\boldsymbol{\theta})$ in (8.16) may be nonzero.

Example 8.7. In this example, the matrices $b(\boldsymbol{\theta})$ and $g(\mathbf{x})$ have real entries. Let $d = 2$, $n = 2$, and $m = 3$. For simplicity, assume that $\Gamma = (2\pi\mathbb{Z})^2$. Suppose that the operator $b(\mathbf{D})$ and the matrix $g(\mathbf{x})$ are given by

$$b(\mathbf{D}) = \begin{pmatrix} D_1 & 0 \\ \frac{1}{2}D_2 & \frac{1}{2}D_1 \\ 0 & D_2 \end{pmatrix}, \quad g(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_2(x_1) & 0 \\ 0 & 0 & g_3(x_1) \end{pmatrix},$$

where $g_2(x_1)$ and $g_3(x_1)$ are (2π)-periodic bounded and positive definite functions of x_1, and $\overline{g_3} = 1$. It is easy to find the Γ-periodic solution of

problem (8.7):

$$\Lambda(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{22}(x_1) & 0 \end{pmatrix}.$$

Here $\Lambda_{22}(x_1)$ is the (2π)-periodic solution of the problem

$$\frac{1}{2}D_1\Lambda_{22}(x_1) + 1 = \underline{g_2}(g_2(x_1))^{-1}, \quad \int\limits_0^{2\pi} \Lambda_{22}(x_1)\,dx_1 = 0.$$

Obviously, $\Lambda_{22}(x_1)$ is purely imaginary. Then $\widetilde{g}(\mathbf{x}) = \operatorname{diag}\{1, \underline{g_2}, g_3(x_1)\}$, and $g^0 = \operatorname{diag}\{1, \underline{g_2}, 1\}$. The spectral germ $\widehat{S}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^*g^0b(\boldsymbol{\theta})$ is given by

$$\widehat{S}(\boldsymbol{\theta}) = \begin{pmatrix} \theta_1^2 + \frac{1}{4}\theta_2^2\underline{g_2} & \frac{1}{4}\theta_1\theta_2\underline{g_2} \\ \frac{1}{4}\theta_1\theta_2\underline{g_2} & \frac{1}{4}\theta_1^2\underline{g_2} + \theta_2^2 \end{pmatrix}, \quad \boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{S}^1. \tag{8.23}$$

It is easily seen that the matrix (8.23) has a multiple eigenvalue (for some $\boldsymbol{\theta}$) only if $\underline{g_2} = 4$.

So, let $\underline{g_2} = 4$. Then the eigenvalues of the germ

$$\widehat{S}(\boldsymbol{\theta}) = \begin{pmatrix} 1 & \theta_1\theta_2 \\ \theta_1\theta_2 & 1 \end{pmatrix}$$

are $\widehat{\gamma}_1(\boldsymbol{\theta}) = 1 + \theta_1\theta_2$ and $\widehat{\gamma}_2(\boldsymbol{\theta}) = 1 - \theta_1\theta_2$. They coincide at four points $\boldsymbol{\theta}^{(1)} = (0, 1)$, $\boldsymbol{\theta}^{(2)} = (0, -1)$, $\boldsymbol{\theta}^{(3)} = (1, 0)$, $\boldsymbol{\theta}^{(4)} = (-1, 0)$.

Next, we calculate the (3×3)-matrix $L(\boldsymbol{\theta})$ (see (8.19)):

$$L(\boldsymbol{\theta}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta_2\overline{\Lambda_{22}^*g_3} \\ 0 & \theta_2\overline{\Lambda_{22}g_3} & 0 \end{pmatrix}.$$

Hence,

$$\widehat{N}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^*L(\boldsymbol{\theta})b(\boldsymbol{\theta}) = \frac{1}{2}\theta_2^3\begin{pmatrix} 0 & \overline{\Lambda_{22}^*g_3} \\ \overline{\Lambda_{22}g_3} & 0 \end{pmatrix}.$$

Below we assume that $\overline{\Lambda_{22}g_3} \neq 0$. (It is easy to give a concrete example: if $g_2(x_1) = 4(1 + \frac{1}{2}\sin x_1)^{-1}$ and $g_3(x_1) = 1 + \frac{1}{2}\cos x_1$, all the conditions are fulfilled.)

For $\boldsymbol{\theta} \neq \boldsymbol{\theta}^{(j)}$, $j = 1, 2, 3, 4$, we have $\widehat{\gamma}_1(\boldsymbol{\theta}) \neq \widehat{\gamma}_2(\boldsymbol{\theta})$ and then $\widehat{N}(\boldsymbol{\theta}) = \widehat{N}_*(\boldsymbol{\theta}) \neq 0$. At the points $\boldsymbol{\theta}^{(1)}$ and $\boldsymbol{\theta}^{(2)}$ we have

$$\widehat{\gamma}_1(\boldsymbol{\theta}^{(j)}) = \widehat{\gamma}_2(\boldsymbol{\theta}^{(j)}) = 1, \quad \widehat{N}(\boldsymbol{\theta}^{(j)}) = \widehat{N}_0(\boldsymbol{\theta}^{(j)}) \neq 0, \quad j = 1, 2.$$

Obviously, the numbers $\pm\mu$, where $\mu = \frac{1}{2}|\overline{\Lambda_{22}g_3}|$, are the eigenvalues of the operator $\widehat{N}(\boldsymbol{\theta}^{(j)})$ for $j = 1, 2$. In the exansions (8.16) there are nonzero coefficients at t^3:

$$\widehat{\lambda}_1(t, \boldsymbol{\theta}^{(j)}) = t^2 + \mu t^3 + \ldots, \quad \widehat{\lambda}_2(t, \boldsymbol{\theta}^{(j)}) = t^2 - \mu t^3 + \ldots, \quad j = 1, 2.$$

In this case, the embrios $\widehat{\omega}_1(\boldsymbol{\theta}^{(j)})$, $\widehat{\omega}_2(\boldsymbol{\theta}^{(j)})$ in the expansions (8.17) can not be real (see Proposition 8.5).

At the points $\boldsymbol{\theta}^{(3)}$ and $\boldsymbol{\theta}^{(4)}$ the situation is different. We have

$$\widehat{\gamma}_1(\boldsymbol{\theta}^{(j)}) = \widehat{\gamma}_2(\boldsymbol{\theta}^{(j)}) = 1, \quad \widehat{N}(\boldsymbol{\theta}^{(j)}) = 0, \quad j = 3, 4.$$

This example also shows that, though the operator $\widehat{N}(\boldsymbol{\theta})$ is always continuous in $\boldsymbol{\theta}$ (it is a polynomial of the third degree), its "blocks" $\widehat{N}_0(\boldsymbol{\theta})$ and $\widehat{N}_*(\boldsymbol{\theta})$ can be discontinuous: at the points where the branches of the eigenvalues of the germ intersect, $\widehat{N}_0(\boldsymbol{\theta})$ and $\widehat{N}_*(\boldsymbol{\theta})$ may have jumps. Moreover, it may happen that $\widehat{N}_0(\boldsymbol{\theta})$ is not equal to zero only at some isolated points.

§9. Approximation of the smoothed operator $e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})}$

9.1. Approximation of the smoothed operator $e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})}$ for $|\mathbf{k}| \leqslant \widehat{t}^0$.
Consider the operator $\mathcal{H}_0 = -\Delta$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. Under the Gelfand transformation, this operator expands in the direct integral of the operators $\mathcal{H}_0(\mathbf{k})$ acting in $L_2(\Omega; \mathbb{C}^n)$. The operator $\mathcal{H}_0(\mathbf{k})$ is given by the differential expression $|\mathbf{D} + \mathbf{k}|^2$ with periodic boundary conditions. Denote

$$\mathcal{R}(\mathbf{k}, \varepsilon) := \varepsilon^2 (\mathcal{H}_0(\mathbf{k}) + \varepsilon^2 I)^{-1}. \tag{9.1}$$

Obviously,

$$\mathcal{R}(\mathbf{k}, \varepsilon)^{s/2} \widehat{P} = \varepsilon^s (t^2 + \varepsilon^2)^{-s/2} \widehat{P}, \quad s > 0. \tag{9.2}$$

We will apply theorems of §4 to the operator $\widehat{A}(t, \boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$. We start with Theorem 4.1. First, we need to specify the constants. By (8.4) and (8.6), instead of the precise values of the constants $\widehat{C}_1(\boldsymbol{\theta}) = \beta_1 \widehat{\delta}^{-1/2} \|\widehat{X}_1(\boldsymbol{\theta})\|$, $\widehat{C}_2(\boldsymbol{\theta}) = \beta_2 \widehat{\delta}^{-1/2} \|\widehat{X}_1(\boldsymbol{\theta})\|^3$ (which depend on $\boldsymbol{\theta}$) we can take

$$\begin{aligned}
\widehat{C}_1 &= 2\beta_1 r_0^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2}, \\
\widehat{C}_2 &= 2\beta_2 r_0^{-1} \alpha_1^{3/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{3/2} \|g^{-1}\|_{L_\infty}^{1/2}.
\end{aligned} \tag{9.3}$$

Combining (4.1), (8.12), and (9.2), we arrive at the inequality

$$\left\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k}, \varepsilon)^{3/2} \widehat{P} \right\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant (\widehat{C}_1 + \widehat{C}_2 |\tau|) \varepsilon,$$

$$\tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |\mathbf{k}| \leqslant \widehat{t}^0. \tag{9.4}$$

9.2. Estimate for $|\mathbf{k}| > \widehat{t}^0$.
For $\mathbf{k} \in \widetilde{\Omega}$ and $|\mathbf{k}| > \widehat{t}^0$ estimates are trivial. By (9.2), we have

$$\|\mathcal{R}(\mathbf{k}, \varepsilon)^{1/2} \widehat{P}\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant (\widehat{t}^0)^{-1} \varepsilon, \quad \varepsilon > 0, \quad \mathbf{k} \in \widetilde{\Omega}, \quad |\mathbf{k}| > \widehat{t}^0. \tag{9.5}$$

Therefore,

$$\left\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k}, \varepsilon)^{1/2} \widehat{P} \right\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant 2(\widehat{t}^0)^{-1} \varepsilon,$$

$$\tau \in \mathbb{R}, \quad \varepsilon > 0, \quad \mathbf{k} \in \widetilde{\Omega}, \quad |\mathbf{k}| > \widehat{t}^0. \tag{9.6}$$

Note that here the smoothing operator is $\mathcal{R}(\mathbf{k}, \varepsilon)^{1/2}$ (i. e., $s = 1$). Of course, the left-hand side of (9.4) also satisfies the same estimate.

From (9.4) and (9.6), using expressions for $\widehat{t}^0$ and $\widehat{C}_1$ (see (8.5) and (9.3)), we obtain

$$\left\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k},\varepsilon)^{3/2}\widehat{P} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant (\widehat{C}_1^* + \widehat{C}_2|\tau|)\varepsilon,$$

$$\tau \in \mathbb{R}, \quad \varepsilon > 0, \quad \mathbf{k} \in \widetilde{\Omega},$$

(9.7)

where $\widehat{C}_1^* = \max\{\widehat{C}_1, 2(\widehat{t}^0)^{-1}\} = \beta_1^* r_0^{-1}\alpha_1^{1/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}$.

9.3. Removal of the operator $\widehat{P}$. Now we show that, up to an admissible error, the projection $\widehat{P}$ can be replaced by the identity operator under the norm sign in (9.7). For this, we estimate the norm of the operator $\mathcal{R}(\mathbf{k},\varepsilon)^{s/2}(I - \widehat{P})$. Under the discrete Fourier transformation (see (6.6)), the operator $\mathcal{R}(\mathbf{k},\varepsilon)^{s/2}$ turns into multiplication of the Fourier coefficients by the symbol $\varepsilon^s(|\mathbf{b}+\mathbf{k}|^2 + \varepsilon^2)^{-s/2}$. The operator $I - \widehat{P}$ makes the zero Fourier coefficient equal to zero. Therefore,

$$\|\mathcal{R}(\mathbf{k},\varepsilon)^{s/2}(I - \widehat{P})\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant \sup_{0\neq\mathbf{b}\in\widetilde{\Gamma}} \varepsilon^s(|\mathbf{b}+\mathbf{k}|^2 + \varepsilon^2)^{-s/2} \leqslant r_0^{-s}\varepsilon^s,$$

$$\varepsilon > 0, \quad \mathbf{k} \in \widetilde{\Omega}.$$

(9.8)

Hence,

$$\left\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k},\varepsilon)^{1/2}(I - \widehat{P}) \right\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant 2r_0^{-1}\varepsilon,$$

$$\tau \in \mathbb{R}, \quad \varepsilon > 0, \quad \mathbf{k} \in \widetilde{\Omega}.$$

(9.9)

Note that here the smoothing operator is $\mathcal{R}(\mathbf{k},\varepsilon)^{1/2}$ (i. e., $s = 1$).

Finally, from (9.7) and (9.9), using the obvious inequality $\|\mathcal{R}(\mathbf{k},\varepsilon)\| \leqslant 1$ and expressions for the constants, we obtain the following result which has been proved before in [BSu5, Theorem 7.1].

Theorem 9.1. *For $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $\mathbf{k} \in \widetilde{\Omega}$ we have*

$$\left\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k},\varepsilon)^{3/2} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant (\widehat{\mathcal{C}}_1 + \widehat{\mathcal{C}}_2|\tau|)\varepsilon,$$

where

$$\widehat{\mathcal{C}}_1 = \widehat{\beta}_1 r_0^{-1}\alpha_1^{1/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2},$$
$$\widehat{\mathcal{C}}_2 = \widehat{C}_2 = 2\beta_2 r_0^{-1}\alpha_1^{3/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}.$$

(9.10)

9.4. Refinement of approximation of the smoothed operator $e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})}$ in the case where $\widehat{N}(\boldsymbol{\theta}) = 0$. Now we apply Theorem 4.2, assuming that $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Taking (8.12) and (9.2) into account, we have

$$\left\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k},\varepsilon)\widehat{P} \right\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant (\widehat{C}_1' + \widehat{C}_5|\tau|)\varepsilon,$$

$$\tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |\mathbf{k}| \leqslant \widehat{t}^0.$$

Here $\widehat{C}'_1 = \max\{2, \widehat{C}_1\}$, and the constant $\widehat{C}_5$ is given by $\widehat{C}_5 = 4\beta_5 r_0^{-2}\alpha_1^2\alpha_0^{-1}\|g\|_{L_\infty}^2\|g^{-1}\|_\infty$.

Together with (9.6) and (9.9) this implies the following result.

Theorem 9.2. *Let $\widehat{N}(\boldsymbol{\theta})$ be the operator defined by (8.18), (8.19). Suppose that $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $\mathbf{k} \in \widetilde{\Omega}$ we have*

$$\left\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k},\varepsilon) \right\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant (\widehat{\mathcal{C}}_3 + \widehat{\mathcal{C}}_4|\tau|)\varepsilon,$$

where $\widehat{\mathcal{C}}_3 = \max\{2 + 2r_0^{-1}, \widehat{\mathcal{C}}_1\}$, $\widehat{\mathcal{C}}_1$ is defined by (9.10), and $\widehat{\mathcal{C}}_4 = \widehat{C}_5 = 4\beta_5 r_0^{-2}\alpha_1^2\alpha_0^{-1}\|g\|_{L_\infty}^2\|g^{-1}\|_\infty$.

Recall that some sufficient conditions ensuring that $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ are given in Proposition 8.4.

9.5. Refinement of approximation of the smoothed operator $e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})}$ in the case where $\widehat{N}_0(\boldsymbol{\theta}) = 0$. Now, we reject the assumption of Theorem 9.2, but instead we assume that $\widehat{N}_0(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta}$. We may also assume that $\widehat{N}(\boldsymbol{\theta}) = \widehat{N}_*(\boldsymbol{\theta}) \neq 0$ for some $\boldsymbol{\theta}$, and then at most points $\boldsymbol{\theta}$ (otherwise, one can apply Theorem 9.2.) We would like to apply the "abstract" result, namely, Theorem 4.3. However, there is an additional difficulty related to the fact that the multiplicities of the eigenvalues of the germ $\widehat{S}(\boldsymbol{\theta})$ may change at some points $\boldsymbol{\theta}$. Near such points the distance between some pair of different eigenvalues tends to zero, and we are not able to choose the parameters (2.3) and (2.4) to be independent of $\boldsymbol{\theta}$. Therefore, we are forced to impose an additional condition. We have to take care only about those pairs of eigenvalues for which the corresponding term in (8.22) is not zero. Since the number of different eigenvalues of the germ and their multiplicities may depend on $\boldsymbol{\theta}$, now it is more convenient to use the initial enumeration of the eigenvalues $\widehat{\gamma}_1(\boldsymbol{\theta}), \ldots, \widehat{\gamma}_n(\boldsymbol{\theta})$ of $\widehat{S}(\boldsymbol{\theta})$ (each eigenvalue is repeated according to its multiplicity). We enumerate them in the nondecreasing order:

$$\widehat{\gamma}_1(\boldsymbol{\theta}) \leqslant \widehat{\gamma}_2(\boldsymbol{\theta}) \leqslant \ldots \leqslant \widehat{\gamma}_n(\boldsymbol{\theta}).$$

For each $\boldsymbol{\theta}$ denote by $\widehat{P}^{(k)}(\boldsymbol{\theta})$ the orthogonal projection of $L_2(\Omega; \mathbb{C}^n)$ onto the eigenspace of $\widehat{S}(\boldsymbol{\theta})$ corresponding to the eigenvalue $\widehat{\gamma}_k(\boldsymbol{\theta})$. Clearly, for each $\boldsymbol{\theta}$ the operator $\widehat{P}^{(k)}(\boldsymbol{\theta})$ coincides with one of the projections $\widehat{P}_j(\boldsymbol{\theta})$ introduced in Subsection 8.7 (but the number j may depend on $\boldsymbol{\theta}$).

Condition 9.3. 1°. *The operator $\widehat{N}_0(\boldsymbol{\theta})$ defined by (8.21) is equal to zero: $\widehat{N}_0(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*
2°. *For any pair of indices (k, r), $1 \leqslant k, r \leqslant n$, $k \neq r$, such that $\widehat{\gamma}_k(\boldsymbol{\theta}_0) = \widehat{\gamma}_r(\boldsymbol{\theta}_0)$ for some $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$, we have $\widehat{P}^{(k)}(\boldsymbol{\theta})\widehat{N}(\boldsymbol{\theta})\widehat{P}^{(r)}(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*

Note that $\widehat{P}^{(k)}(\boldsymbol{\theta}_0) = \widehat{P}^{(r)}(\boldsymbol{\theta}_0)$ at the points $\boldsymbol{\theta}_0$ such that $\widehat{\gamma}_k(\boldsymbol{\theta}_0) = \widehat{\gamma}_r(\boldsymbol{\theta}_0)$. Therefore, the identity $\widehat{P}^{(k)}(\boldsymbol{\theta}_0)\widehat{N}(\boldsymbol{\theta}_0)\widehat{P}^{(r)}(\boldsymbol{\theta}_0) = 0$ holds automatically in

virtue of condition $1°$. Condition $2°$ can be reformulated as follows: we assume that, for the "blocks" $\widehat{P}^{(k)}(\boldsymbol{\theta})\widehat{N}(\boldsymbol{\theta})\widehat{P}^{(r)}(\boldsymbol{\theta})$ of the operator $\widehat{N}(\boldsymbol{\theta})$ that are not identically zero, the corresponding branches of the eigenvalues $\widehat{\gamma}_k(\boldsymbol{\theta})$ and $\widehat{\gamma}_r(\boldsymbol{\theta})$ do not intersect.

Obviously, Condition 9.3 is ensured by the following more restrictive condition.

Condition 9.4. $1°$. *The operator $\widehat{N}_0(\boldsymbol{\theta})$ defined by (8.21) is equal to zero*: $\widehat{N}_0(\boldsymbol{\theta}) = 0$ *for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*
$2°$. *Assume that the number p of different eigenvalues of the spectral germ $\widehat{S}(\boldsymbol{\theta})$ does not depend on $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Denote different eigenvalues of the germ enumerated in the increasing order by $\widehat{\gamma}_1^{\circ}(\boldsymbol{\theta}), \ldots, \widehat{\gamma}_p^{\circ}(\boldsymbol{\theta})$, and assume that their multiplicities $k_1, \ldots, k_p$ do not depend on $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*

Remark 9.5. Assumption $2°$ of Condition 9.4 is a fortiori satisfied, if the spectrum of the germ $\widehat{S}(\boldsymbol{\theta})$ is simple for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

So, we assume that Condition 9.3 is satisfied. We are interested only in the pairs of indices from the set

$$\widehat{\mathcal{K}} := \{(k, r) : \ 1 \leqslant k, r \leqslant n, \ k \neq r, \ \widehat{P}^{(k)}(\boldsymbol{\theta})\widehat{N}(\boldsymbol{\theta})\widehat{P}^{(r)}(\boldsymbol{\theta}) \not\equiv 0\}.$$

Denote (cf. (2.3))

$$\widehat{c}_{kr}^{\circ}(\boldsymbol{\theta}) := \min\{\widehat{c}_*, n^{-1}|\widehat{\gamma}_k(\boldsymbol{\theta}) - \widehat{\gamma}_r(\boldsymbol{\theta})|\}, \quad (k, r) \in \widehat{\mathcal{K}}.$$

Since $\widehat{S}(\boldsymbol{\theta})$ is continuous in $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ (this is a polynomial of the second degree), then the perturbation theory of discrete spectrum implies that the functions $\widehat{\gamma}_j(\boldsymbol{\theta})$ are continuous on the sphere $\mathbb{S}^{d-1}$. By Condition $9.3(2°)$, for $(k, r) \in \widehat{\mathcal{K}}$ we have $|\widehat{\gamma}_k(\boldsymbol{\theta}) - \widehat{\gamma}_r(\boldsymbol{\theta})| > 0$ for all $\boldsymbol{\theta}$, whence

$$\widehat{c}_{kr}^{\circ} := \min_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} \widehat{c}_{kr}^{\circ}(\boldsymbol{\theta}) > 0, \quad (k, r) \in \widehat{\mathcal{K}}.$$

We put

$$\widehat{c}^{\circ} := \min_{(k,r) \in \widehat{\mathcal{K}}} \widehat{c}_{kr}^{\circ}. \tag{9.11}$$

Clearly, the number (9.11) is a realization of (3.33) chosen independently of $\boldsymbol{\theta}$.

Under Condition 9.3, the number t^{00} subject to (3.34) also can be chosen independently of $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Taking (8.4) and (8.6) into account, we put

$$\widehat{t}^{00} = (8\beta_2)^{-1} r_0 \alpha_1^{-3/2} \alpha_0^{1/2} \|g\|_{L_\infty}^{-3/2} \|g^{-1}\|_{L_\infty}^{-1/2} \widehat{c}^{\circ}, \tag{9.12}$$

where $\widehat{c}^{\circ}$ is defined by (9.11). (The condition $\widehat{t}^{00} \leqslant \widehat{t}^0$ is valid automatically since $\widehat{c}^{\circ} \leqslant \|\widehat{S}(\boldsymbol{\theta})\| \leqslant \alpha_1 \|g\|_{L_\infty}$.)

Remark 9.6. 1. Unlike $\widehat{t}^0$ (see (8.5)) that is controlled only in terms of r_0, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$, the number $\widehat{t}^{00}$ depends on the spectral characteristics of the germ, namely, on the minimal distance between its different eigenvalues $\widehat{\gamma}_k(\boldsymbol{\theta})$ and $\widehat{\gamma}_r(\boldsymbol{\theta})$ (where (k, r) runs through $\widehat{\mathcal{K}}$). 2. If we reject Condition 9.3 and admit intersection of the branches $\widehat{\gamma}_k(\boldsymbol{\theta})$ and $\widehat{\gamma}_r(\boldsymbol{\theta})$

(for some $(k, r) \in \widehat{\mathcal{K}}$), then $\widehat{c}^{\circ}_{kr}(\boldsymbol{\theta})$ will be not positive definite, and we will be not able to choose the number $\widehat{t}^{00}$ independently of $\boldsymbol{\theta}$.

Under Condition 9.3, we apply Theorem 4.3 and obtain

$$\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k}, \varepsilon)\widehat{P}\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant (\widehat{C}'_9 + \widehat{C}_{10}|\tau|)\varepsilon,$$
$$\tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |\mathbf{k}| \leqslant \widehat{t}^{00}, \tag{9.13}$$

where

$$\widehat{C}'_9 = \max\{2, \beta_9 r_0^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} \left(1 + n^2 \alpha_1 \|g\|_{L_\infty} (\widehat{c}^{\circ})^{-1}\right)\},$$
$$\widehat{C}_{10} = \beta_{10} r_0^{-2} \alpha_1^2 \alpha_0^{-1} \|g\|_{L_\infty}^2 \|g^{-1}\|_{L_\infty} \left(1 + n^2 \alpha_1^2 \|g\|_{L_\infty}^2 (\widehat{c}^{\circ})^{-2}\right). \tag{9.14}$$

Similarly to (9.6), we have

$$\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k}, \varepsilon)^{1/2}\widehat{P}\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant 2(\widehat{t}^{00})^{-1}\varepsilon,$$
$$\tau \in \mathbb{R}, \quad \varepsilon > 0, \quad \mathbf{k} \in \widetilde{\Omega}, \quad |\mathbf{k}| > \widehat{t}^{00}. \tag{9.15}$$

Now, relations (9.9), (9.13), and (9.15) directly imply the following result.

Theorem 9.7. *Suppose that Condition 9.3 (or more restrictive Condition 9.4) is satisfied. Then for any $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $\mathbf{k} \in \widetilde{\Omega}$ we have*

$$\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k}, \varepsilon)\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant (\widehat{\mathcal{C}}_5 + \widehat{\mathcal{C}}_6|\tau|)\varepsilon, \quad (9.16)$$

where $\widehat{\mathcal{C}}_5 = \max\{\widehat{C}'_9, 2(\widehat{t}^{00})^{-1}\} + 2r_0^{-1}$, $\widehat{\mathcal{C}}_6 = \widehat{C}_{10}$, and the constants $\widehat{C}'_9$, $\widehat{C}_{10}$, and $\widehat{t}^{00}$ are defined by (9.14) and (9.12).

The assumptions of Theorem 9.7 are a fortiori satisfied in the "real" case, if the spectrum of the germ is simple (see Corollary 8.6 and Remark 9.5). We arrive at the following corollary.

Corollary 9.8. *Suppose that the matrices $b(\boldsymbol{\theta})$ and $g(\mathbf{x})$ have real entries. Suppose that the spectrum of the germ $\widehat{S}(\boldsymbol{\theta})$ is simple for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then estimate (9.16) holds for any $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $\mathbf{k} \in \widetilde{\Omega}$.*

9.6. The sharpness of the result in the general case. Application of Theorem 4.4 allows us to confirm the sharpness of the result of Theorem 9.1 in the general case.

Theorem 9.9. *Let $\widehat{N}_0(\boldsymbol{\theta})$ be the operator defined by (8.21). Suppose that $\widehat{N}_0(\boldsymbol{\theta}_0) \neq 0$ at some point $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$. Let $0 \neq \tau \in \mathbb{R}$. Then for any $1 \leqslant s < 3$ it is impossible that the estimate*

$$\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k}, \varepsilon)^{s/2}\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant \mathcal{C}(\tau)\varepsilon \tag{9.17}$$

holds for almost all $\mathbf{k} = t\boldsymbol{\theta} \in \widetilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

For the proof we need the following lemma.

Lemma 9.10. *Let $\widehat{\delta}$ and $\widehat{t}^0$ be given by (8.4) and (8.5), respectively. Let $\widehat{F}(\mathbf{k}) = \widehat{F}(t, \boldsymbol{\theta})$ be the spectral projection of the operator $\widehat{\mathcal{A}}(\mathbf{k})$ for the interval $[0, \widehat{\delta}]$. Then for $|\mathbf{k}| \leqslant \widehat{t}^0$ and $|\mathbf{k}_0| \leqslant \widehat{t}^0$ we have*

$$\|\widehat{F}(\mathbf{k}) - \widehat{F}(\mathbf{k}_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant \widehat{C}'|\mathbf{k} - \mathbf{k}_0|,$$
$$\|\widehat{\mathcal{A}}(\mathbf{k})\widehat{F}(\mathbf{k}) - \widehat{\mathcal{A}}(\mathbf{k}_0)\widehat{F}(\mathbf{k}_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant \widehat{C}''|\mathbf{k} - \mathbf{k}_0|,$$
$$\|e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k})}\widehat{F}(\mathbf{k}) - e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k}_0)}\widehat{F}(\mathbf{k}_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant (2\widehat{C}' + \widehat{C}''|\tau|)|\mathbf{k} - \mathbf{k}_0|.$$
$$\tag{9.18}$$

Proof. First, we estimate the difference of the resolvents of the operators $\widehat{\mathcal{A}}(\mathbf{k})$ and $\widehat{\mathcal{A}}(\mathbf{k}_0)$. Consider the difference of the corresponding sesquilinear forms on the elements $\mathbf{u}, \mathbf{v} \in \widetilde{H}^1(\Omega; \mathbb{C}^n)$:

$$\widehat{a}(\mathbf{k})[\mathbf{u}, \mathbf{v}] - \widehat{a}(\mathbf{k}_0)[\mathbf{u}, \mathbf{v}]$$
$$= \int_\Omega \left(\langle g(\mathbf{x})b(\mathbf{k} - \mathbf{k}_0)\mathbf{u}, b(\mathbf{D} + \mathbf{k})\mathbf{v} \rangle + \langle g(\mathbf{x})b(\mathbf{D} + \mathbf{k}_0)\mathbf{u}, b(\mathbf{k} - \mathbf{k}_0)\mathbf{v} \rangle \right) \, d\mathbf{x}.$$

Let $z \in \mathbb{C}$ be a common regular point of $\widehat{\mathcal{A}}(\mathbf{k})$ and $\widehat{\mathcal{A}}(\mathbf{k}_0)$. Substituting $\mathbf{u} = (\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1}\boldsymbol{\varphi}$ and $\mathbf{v} = (\widehat{\mathcal{A}}(\mathbf{k}_0) - z^*I)^{-1}\boldsymbol{\psi}$, where $\boldsymbol{\varphi}, \boldsymbol{\psi} \in L_2(\Omega; \mathbb{C}^n)$, it is easy to see that

$$\left| ((\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1}\boldsymbol{\varphi} - (\widehat{\mathcal{A}}(\mathbf{k}_0) - zI)^{-1}\boldsymbol{\varphi}, \boldsymbol{\psi})_{L_2(\Omega)} \right|$$
$$\leqslant \|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}|\mathbf{k} - \mathbf{k}_0| \|(\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1}\boldsymbol{\varphi}\|_{L_2} \|\widehat{\mathcal{A}}(\mathbf{k}_0)^{1/2}(\widehat{\mathcal{A}}(\mathbf{k}_0) - z^*I)^{-1}\boldsymbol{\psi}\|_{L_2}$$
$$+ \|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}|\mathbf{k} - \mathbf{k}_0| \|(\widehat{\mathcal{A}}(\mathbf{k}_0) - z^*I)^{-1}\boldsymbol{\psi}\|_{L_2} \|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1}\boldsymbol{\varphi}\|_{L_2}$$
$$+ 2\|g\|_{L_\infty}\alpha_1|\mathbf{k} - \mathbf{k}_0|^2 \|(\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1}\boldsymbol{\varphi}\|_{L_2} \|(\widehat{\mathcal{A}}(\mathbf{k}_0) - z^*I)^{-1}\boldsymbol{\psi}\|_{L_2}.$$
$$\tag{9.19}$$

As follows from the results of §1, for $|\mathbf{k}| \leqslant \widehat{t}^0$ the first n eigenvalues of the operator $\widehat{\mathcal{A}}(\mathbf{k})$ lie on the interval $[0, \widehat{\delta}]$, and the rest of the spectrum lies on the semiaxis $[3\widehat{\delta}, \infty)$. Consider the contour $\Gamma_{\widehat{\delta}} \subset \mathbb{C}$ which encloses the interval $[0, \widehat{\delta}]$ equidistantly at the distance $\widehat{\delta}$. Then

$$\widehat{F}(\mathbf{k}) = -\frac{1}{2\pi i} \int_{\Gamma_{\widehat{\delta}}} (\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1} \, dz, \tag{9.20}$$

where we integrate in the positive direction. If $z \in \Gamma_{\widehat{\delta}}$, then

$$\|(\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1}\boldsymbol{\varphi}\|_{L_2(\Omega)} \leqslant \widehat{\delta}^{-1}\|\boldsymbol{\varphi}\|_{L_2(\Omega)},$$
$$\|(\widehat{\mathcal{A}}(\mathbf{k}_0) - z^*I)^{-1}\boldsymbol{\psi}\|_{L_2(\Omega)} \leqslant \widehat{\delta}^{-1}\|\boldsymbol{\psi}\|_{L_2(\Omega)},$$
$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1}\boldsymbol{\varphi}\|_{L_2(\Omega)} \leqslant \sqrt{3}\,\widehat{\delta}^{-1/2}\|\boldsymbol{\varphi}\|_{L_2(\Omega)},$$
$$\|\widehat{\mathcal{A}}(\mathbf{k}_0)^{1/2}(\widehat{\mathcal{A}}(\mathbf{k}_0) - z^*I)^{-1}\boldsymbol{\psi}\|_{L_2(\Omega)} \leqslant \sqrt{3}\,\widehat{\delta}^{-1/2}\|\boldsymbol{\psi}\|_{L_2(\Omega)}.$$
$$\tag{9.21}$$

From (9.19) and (9.21) it follows that

$$\|(\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1} - (\widehat{\mathcal{A}}(\mathbf{k}_0) - zI)^{-1}\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant C|\mathbf{k} - \mathbf{k}_0| \qquad (9.22)$$

for $z \in \Gamma_{\widehat{\delta}}$ and $|\mathbf{k}|, |\mathbf{k}_0| \leqslant \widehat{t}^0$. Now representation (9.20) (at the points $\mathbf{k}$ and $\mathbf{k}_0$) and (9.22) imply the first inequality in (9.18).

The second estimate in (9.18) is deduced from the representation

$$\widehat{\mathcal{A}}(\mathbf{k})\widehat{F}(\mathbf{k}) = -\frac{1}{2\pi i} \int_{\Gamma_{\widehat{\delta}}} z(\widehat{\mathcal{A}}(\mathbf{k}) - zI)^{-1} \, dz \qquad (9.23)$$

at the points $\mathbf{k}$ and $\mathbf{k}_0$ with the help of (9.22).

Let us prove the third inequality. We have

$$e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k})}\widehat{F}(\mathbf{k}) - e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k}_0)}\widehat{F}(\mathbf{k}_0) = e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k})}\widehat{F}(\mathbf{k})(\widehat{F}(\mathbf{k}) - \widehat{F}(\mathbf{k}_0))$$
$$+ (\widehat{F}(\mathbf{k}) - \widehat{F}(\mathbf{k}_0))e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k}_0)}\widehat{F}(\mathbf{k}_0) + \Xi(\tau, \mathbf{k}, \mathbf{k}_0), \qquad (9.24)$$

where

$$\Xi(\tau, \mathbf{k}, \mathbf{k}_0) = e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k})}\widehat{F}(\mathbf{k})\widehat{F}(\mathbf{k}_0) - \widehat{F}(\mathbf{k})e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k}_0)}\widehat{F}(\mathbf{k}_0).$$

The sum of the first two terms in (9.24) does not exceed $2\widehat{C}'|\mathbf{k} - \mathbf{k}_0|$, in view of the first estimate in (9.18). The third term can be written as

$$\Xi(\tau, \mathbf{k}, \mathbf{k}_0) = e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k})}\Sigma(\tau, \mathbf{k}, \mathbf{k}_0),$$
$$\Sigma(\tau, \mathbf{k}, \mathbf{k}_0) = \widehat{F}(\mathbf{k})\widehat{F}(\mathbf{k}_0) - e^{i\tau\widehat{\mathcal{A}}(\mathbf{k})}\widehat{F}(\mathbf{k})e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k}_0)}\widehat{F}(\mathbf{k}_0).$$

Obviously, $\Sigma(0, \mathbf{k}, \mathbf{k}_0) = 0$, and

$$\frac{d\Sigma(\tau, \mathbf{k}, \mathbf{k}_0)}{d\tau} = -i\widehat{F}(\mathbf{k})e^{i\tau\widehat{\mathcal{A}}(\mathbf{k})}(\widehat{\mathcal{A}}(\mathbf{k})\widehat{F}(\mathbf{k}) - \widehat{\mathcal{A}}(\mathbf{k}_0)\widehat{F}(\mathbf{k}_0))e^{-i\tau\widehat{\mathcal{A}}(\mathbf{k}_0)}\widehat{F}(\mathbf{k}_0).$$

Integrating over the interval $[0, \tau]$ and using the second estimate in (9.18), we obtain

$$\|\Xi(\tau, \mathbf{k}, \mathbf{k}_0)\| = \|\Sigma(\tau, \mathbf{k}, \mathbf{k}_0)\| \leqslant \widehat{C}''|\tau||\mathbf{k} - \mathbf{k}_0|.$$

We arrive at the third estimate in (9.18). ●

Proof of Theorem 9.9. We prove by contradiction. Let us fix $\tau \neq 0$. Assume that for some $1 \leqslant s < 3$ there exists a constant $\mathcal{C}(\tau) > 0$ such that estimate (9.17) holds for almost every $\mathbf{k} \in \widetilde{\Omega}$ and sufficiently small $\varepsilon > 0$. By (9.9) and (9.2), it follows that there exists a constant $\widetilde{\mathcal{C}}(\tau) > 0$ such that

$$\left\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})}\right)\widehat{P}\right\|_{L_2(\Omega) \to L_2(\Omega)}\varepsilon^s(|\mathbf{k}|^2 + \varepsilon^2)^{-s/2} \leqslant \widetilde{\mathcal{C}}(\tau)\varepsilon \tag{9.25}$$

for almost every $\mathbf{k} \in \widetilde{\Omega}$ and sufficiently small ε.

Now, let $|\mathbf{k}| \leqslant \widehat{t}^0$. By (1.13),

$$\|\widehat{F}(\mathbf{k}) - \widehat{P}\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant \widehat{C}_1|\mathbf{k}|, \quad |\mathbf{k}| \leqslant \widehat{t}^0. \tag{9.26}$$

From (9.25) and (9.26) it follows that there exists a constant $\check{\mathcal{C}}(\tau) > 0$ such that

$$\|e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(\mathbf{k})}\widehat{F}(\mathbf{k}) - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(\mathbf{k})}\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)}\varepsilon^s(|\mathbf{k}|^2+\varepsilon^2)^{-s/2} \leqslant \check{\mathcal{C}}(\tau)\varepsilon \tag{9.27}$$

for almost every $\mathbf{k}$ in the ball $|\mathbf{k}| \leqslant \widehat{t}^0$ and sufficiently small ε.

Observe that $\widehat{P}$ is the spectral projection of the operator $\widehat{\mathcal{A}}^0(\mathbf{k})$ for the interval $[0, \widehat{\delta}]$. Applying Lemma 9.10 to $\widehat{\mathcal{A}}(\mathbf{k})$ and $\widehat{\mathcal{A}}^0(\mathbf{k})$, we conclude that for fixed τ and ε the operator under the norm sign in (9.27) is continuous with respect to $\mathbf{k}$ in the ball $|\mathbf{k}| \leqslant \widehat{t}^0$. Consequently, estimate (9.27) holds for all $\mathbf{k}$ in that ball. In particular, it holds at the point $\mathbf{k} = t\boldsymbol{\theta}_0$ if $t \leqslant \widehat{t}^0$. Applying (9.26) once more, we see that

$$\| \left(e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}(t\boldsymbol{\theta}_0)} - e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}^0(t\boldsymbol{\theta}_0)}\right)\widehat{P}\|_{L_2(\Omega)\to L_2(\Omega)}\varepsilon^s(t^2+\varepsilon^2)^{-s/2} \leqslant \check{\mathcal{C}}'(\tau)\varepsilon \tag{9.28}$$

for all $t \leqslant \widehat{t}^0$ and sufficiently small ε.

Estimate (9.28) corresponds to the abstract estimate (4.3). Since $\widehat{N}_0(\boldsymbol{\theta}_0) \neq 0$, applying Theorem 4.4, we arrive at a contradiction. $\bullet$

§10. The operator $\mathcal{A}(\mathbf{k})$. Application of the scheme of §5

10.1. Application of the scheme of §5 to the operator $\mathcal{A}(\mathbf{k})$. We apply the scheme of §5 to study the operator $\mathcal{A}(\mathbf{k}) = f^*\widehat{\mathcal{A}}(\mathbf{k})f$. Now $\mathfrak{H} = \widehat{\mathfrak{H}} = L_2(\Omega;\mathbb{C}^n)$, $\mathfrak{H}_* = L_2(\Omega;\mathbb{C}^m)$, the role of $A(t)$ is played by $A(t,\boldsymbol{\theta}) = \mathcal{A}(\mathbf{k})$, the role of $\widehat{A}(t)$ is played by $\widehat{A}(t,\boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$. Next, the isomorphism M is the operator of multiplication by the matrix-valued function $f(\mathbf{x})$. The operator Q is the operator of multiplication by the matrix-valued function

$$Q(\mathbf{x}) = (f(\mathbf{x})f(\mathbf{x})^*)^{-1}.$$

The block of the operator Q in the subspace $\widehat{\mathfrak{N}}$ (see (8.2)) is the operator of multiplication by the constant matrix

$$\overline{Q} = \underline{(ff^*)}^{-1} = |\Omega|^{-1}\int_\Omega (f(\mathbf{x})f(\mathbf{x})^*)^{-1}\,d\mathbf{x}.$$

Next, M_0 is the operator of multiplication by the constant matrix

$$f_0 = (\overline{Q})^{-1/2} = \underline{(ff^*)}^{1/2}. \tag{10.1}$$

Note that

$$|f_0| \leqslant \|f\|_{L_\infty}, \quad |f_0^{-1}| \leqslant \|f^{-1}\|_{L_\infty}. \tag{10.2}$$

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we define the operator

$$\mathcal{A}^0 := f_0\widehat{\mathcal{A}}^0 f_0 = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D})f_0. \tag{10.3}$$

Let $\mathcal{A}^0(\mathbf{k})$ be the corresponding family of operators in $L_2(\Omega;\mathbb{C}^n)$. Then $\mathcal{A}^0(\mathbf{k}) = f_0\widehat{\mathcal{A}}^0(\mathbf{k})f_0$. By (8.3) and (8.10), we have

$$f_0\widehat{S}(\mathbf{k})f_0\widehat{P} = \mathcal{A}^0(\mathbf{k})\widehat{P}. \tag{10.4}$$

10.2. The analytic branches of eigenvalues and eigenvectors. According to (5.3), the spectral germ $S(\boldsymbol{\theta})$ of the operator $A(t,\boldsymbol{\theta})$ acting in the subspace $\mathfrak{N}$ (see (7.3)) is represented as

$$S(\boldsymbol{\theta}) = Pf^*b(\boldsymbol{\theta})^*g^0b(\boldsymbol{\theta})f|_{\mathfrak{N}},$$

where P is the orthogonal projection of $L_2(\Omega;\mathbb{C}^n)$ onto $\mathfrak{N}$.

The analytic (in t) branches of the eigenvalues $\lambda_l(t,\boldsymbol{\theta})$ and the branches of the eigenvectors $\varphi_l(t,\boldsymbol{\theta})$ of $A(t,\boldsymbol{\theta})$ admit the power series expansions of the form (1.6), (1.7) with the coefficients depending on $\boldsymbol{\theta}$:

$$\lambda_l(t,\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \mu_l(\boldsymbol{\theta})t^3 + \dots, \quad l = 1,\dots,n, \tag{10.5}$$

$$\varphi_l(t,\boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta})t^2 + t\psi_l^{(1)}(\boldsymbol{\theta}) + \dots, \quad l = 1,\dots,n. \tag{10.6}$$

The vectors $\omega_1(\boldsymbol{\theta}),\dots,\omega_n(\boldsymbol{\theta})$ form an orthonormal basis in the subspace $\mathfrak{N}$ (see (7.3)), and the vectors

$$\zeta_l(\boldsymbol{\theta}) := f\omega_l(\boldsymbol{\theta}), \quad l = 1,\dots,n,$$

form a basis in $\widehat{\mathfrak{N}}$ (see (8.2)) orthonormal with the weight $\overline{Q}$, i. e., $(\overline{Q}\zeta_l(\boldsymbol{\theta}),\zeta_j(\boldsymbol{\theta})) = \delta_{jl}$, $j,l = 1,\dots,n$.

The numbers $\gamma_l(\boldsymbol{\theta})$ and the elements $\omega_l(\boldsymbol{\theta})$ are eigenvalues and eigenvectors of the spectral germ $S(\boldsymbol{\theta})$. However, it is more convenient to turn to the generalized spectral problem for $\widehat{S}(\boldsymbol{\theta})$. According to (5.7), the numbers $\gamma_l(\boldsymbol{\theta})$ and the elements $\zeta_l(\boldsymbol{\theta})$ are eigenvalues and eigenvectors of the following generalized spectral problem:

$$b(\boldsymbol{\theta})^*g^0b(\boldsymbol{\theta})\zeta_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})\overline{Q}\zeta_l(\boldsymbol{\theta}), \quad l = 1,\dots,n. \tag{10.7}$$

10.3. The operator $\widehat{N}_Q(\boldsymbol{\theta})$. We need to describe the operator $\widehat{N}_Q(\boldsymbol{\theta})$ (in abstract terms it was defined in Subsection 5.2). Let $\Lambda_Q(\mathbf{x})$ be the Γ-periodic solution of the problem

$$b(\mathbf{D})^*g(\mathbf{x})(b(\mathbf{D})\Lambda_Q(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega Q(\mathbf{x})\Lambda_Q(\mathbf{x})\,d\mathbf{x} = 0. \tag{10.8}$$

Clearly, $\Lambda_Q(\mathbf{x})$ differs from the periodic solution $\Lambda(\mathbf{x})$ of the problem (8.7) by a constant summand:

$$\Lambda_Q(\mathbf{x}) = \Lambda(\mathbf{x}) + \Lambda_Q^0, \quad \Lambda_Q^0 = -(\overline{Q})^{-1}(\overline{Q\Lambda}). \tag{10.9}$$

As shown in [BSu3,§5], the operator $\widehat{N}_Q(\boldsymbol{\theta})$ takes the form

$$\widehat{N}_Q(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^*L_Q(\boldsymbol{\theta})b(\boldsymbol{\theta})\widehat{P}, \tag{10.10}$$

where $L_Q(\boldsymbol{\theta})$ is an $(m \times m)$-matrix given by

$$L_Q(\boldsymbol{\theta}) = |\Omega|^{-1}\int_\Omega (\Lambda_Q(\mathbf{x})^*b(\boldsymbol{\theta})^*\widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^*b(\boldsymbol{\theta})\Lambda_Q(\mathbf{x}))\,d\mathbf{x}. \tag{10.11}$$

Combining (10.9), (10.11), and (8.19), we see that

$$L_Q(\boldsymbol{\theta}) = L(\boldsymbol{\theta}) + L_Q^0(\boldsymbol{\theta}), \quad L_Q^0(\boldsymbol{\theta}) = (\Lambda_Q^0)^*b(\boldsymbol{\theta})^*g^0 + g^0b(\boldsymbol{\theta})\Lambda_Q^0.$$

Observe that $L_Q(\mathbf{k}) := t L_Q(\boldsymbol{\theta})$, $\mathbf{k} \in \mathbb{R}^d$, is a Hermitian matrix-valued function first order homogeneous in $\mathbf{k}$. We put $\widehat{N}_Q(\mathbf{k}) := t^3 \widehat{N}_Q(\boldsymbol{\theta})$, $\mathbf{k} \in \mathbb{R}^d$. Then $\widehat{N}_Q(\mathbf{k}) = b(\mathbf{k})^* L_Q(\mathbf{k}) b(\mathbf{k}) \widehat{P}$. The matrix-valued function $b(\mathbf{k})^* L_Q(\mathbf{k}) b(\mathbf{k})$ is a homogeneous polynomial of the third degree in $\mathbf{k} \in \mathbb{R}^d$. It follows that either $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ identically for $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, or $\widehat{N}_Q(\boldsymbol{\theta}) \neq 0$ for most points $\boldsymbol{\theta}$ (except for the zeroes of this polynomial).

Some cases where the operator (10.10) is equal to zero were distinguished in [BSu3, §5].

Proposition 10.1. *Suppose that at least one of the following conditions is satisfied*:
1°. *The operator $\mathcal{A}$ has the form $\mathcal{A} = f(\mathbf{x})^* \mathbf{D}^* g(\mathbf{x}) \mathbf{D} f(\mathbf{x})$, where $g(\mathbf{x})$ is a symmetric matrix with real entries.*
2°. *Relations (8.14) are satisfied, i. e. $g^0 = \overline{g}$.*
Then $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

Recall that (see Subsections 5.2, 5.3) $\widehat{N}_Q(\boldsymbol{\theta}) = \widehat{N}_{0,Q}(\boldsymbol{\theta}) + \widehat{N}_{*,Q}(\boldsymbol{\theta})$. By (5.9),

$$\widehat{N}_{0,Q}(\boldsymbol{\theta}) = \sum_{l=1}^{n} \mu_l(\boldsymbol{\theta}) (\cdot, \overline{Q} \zeta_l(\boldsymbol{\theta}))_{L_2(\Omega)} \overline{Q} \zeta_l(\boldsymbol{\theta}).$$

We have

$$(\widehat{N}_Q(\boldsymbol{\theta}) \zeta_l(\boldsymbol{\theta}), \zeta_l(\boldsymbol{\theta}))_{L_2(\Omega)} = (\widehat{N}_{0,Q}(\boldsymbol{\theta}) \zeta_l(\boldsymbol{\theta}), \zeta_l(\boldsymbol{\theta}))_{L_2(\Omega)} = \mu_l(\boldsymbol{\theta}), \quad l = 1, \ldots, n.$$
(10.12)

Now we assume that the matrices $b(\boldsymbol{\theta})$, $g(\mathbf{x})$, and $Q(\mathbf{x})$ *have real entries*. Then the matrix $\Lambda_Q(\mathbf{x})$ (see (10.8)) has purely imaginary entries, and $\widetilde{g}(\mathbf{x})$ and g^0 have real entries. In this case $L_Q(\boldsymbol{\theta})$ (see (10.11)) and $b(\boldsymbol{\theta})^* L_Q(\boldsymbol{\theta}) b(\boldsymbol{\theta})$ are Hermitian matrices with purely imaginary entries. If the analytic branches of the eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and the analytic branches of the eigenvectors $\varphi_l(t, \boldsymbol{\theta})$ of the operator $A(t, \boldsymbol{\theta})$ can be chosen so that the vectors $\zeta_l(\boldsymbol{\theta}) = f \omega_l(\boldsymbol{\theta})$, $l = 1, \ldots, n$, are real, then, by (10.12), we have $\mu_l(\boldsymbol{\theta}) = 0$, $l = 1, \ldots, n$, i. e., $\widehat{N}_{0,Q}(\boldsymbol{\theta}) = 0$. We arrive at the following statement.

Proposition 10.2. *Suppose that the matrices $b(\boldsymbol{\theta})$, $g(\mathbf{x})$, and $Q(\mathbf{x})$ have real entries. Suppose that in the expansions (10.6) for the analytic branches of the eigenvectors of $A(t, \boldsymbol{\theta})$ the "embrios" $\omega_l(\boldsymbol{\theta})$, $l = 1, \ldots, n$, can be chosen so that the vectors $\zeta_l(\boldsymbol{\theta}) = f \omega_l(\boldsymbol{\theta})$ are real. Then in (10.5) we have $\mu_l(\boldsymbol{\theta}) = 0$, $l = 1, \ldots, n$, i. e., $\widehat{N}_{0,Q}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*

In the "real" case under consideration, the operator $\widehat{S}(\boldsymbol{\theta})$ is a symmetric matrix with real entries; $\overline{Q}$ is also a symmetric matrix with real entries. Clearly, if the eigenvalue $\gamma_j(\boldsymbol{\theta})$ of the generalized problem (10.7) is simple, then the eigenvector $\zeta_j(\boldsymbol{\theta}) = f \omega_j(\boldsymbol{\theta})$ is defined uniquely up to a phase factor, and we always can choose it to be real. We arrive at the following corollary.

Corollary 10.3. *Suppose that the matrices* $b(\boldsymbol{\theta})$, $g(\mathbf{x})$, *and* $Q(\mathbf{x})$ *have real entries. Suppose that the spectrum of the generalized spectral problem* (10.7) *is simple. Then* $\widehat{N}_{0,Q}(\boldsymbol{\theta}) = 0$ *for all* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

10.4. Multiplicities of the eigenvalues of the germ. This subsection concerns the case where $n \geqslant 2$. We return to the notation of §2, tracing the multiplicities of the eigenvalues of the spectral germ $S(\boldsymbol{\theta})$. From what was said in Subsection 10.2 it follows that these eigenvalues are also the eigenvalues of the generalized problem (10.7). In general, the number $p(\boldsymbol{\theta})$ of different eigenvalues $\gamma_1^{\circ}(\boldsymbol{\theta}), \ldots, \gamma_{p(\boldsymbol{\theta})}^{\circ}(\boldsymbol{\theta})$ of this problem and their multiplicities $k_1(\boldsymbol{\theta}), \ldots, k_{p(\boldsymbol{\theta})}(\boldsymbol{\theta})$ depend on the parameter $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. For a fixed $\boldsymbol{\theta}$, let $\mathfrak{N}_j(\boldsymbol{\theta})$ be the eigenspace of the germ $S(\boldsymbol{\theta})$ corresponding to the eigenvalue $\gamma_j^{\circ}(\boldsymbol{\theta})$. Then $f\mathfrak{N}_j(\boldsymbol{\theta})$ is the eigenspace of the problem (10.7) corresponding to the same eigenvalue $\gamma_j^{\circ}(\boldsymbol{\theta})$. Let $\mathcal{P}_j(\boldsymbol{\theta})$ denote the "skew" projection of $L_2(\Omega; \mathbb{C}^n)$ onto the subspace $f\mathfrak{N}_j(\boldsymbol{\theta})$; $\mathcal{P}_j(\boldsymbol{\theta})$ is orthogonal with respect to the inner product with the weight $\overline{Q}$. Then, by (5.11), we have the following invariant representations for the operators $\widehat{N}_{0,Q}(\boldsymbol{\theta})$ and $\widehat{N}_{*,Q}(\boldsymbol{\theta})$:

$$\widehat{N}_{0,Q}(\boldsymbol{\theta}) = \sum_{j=1}^{p(\boldsymbol{\theta})} \mathcal{P}_j(\boldsymbol{\theta})^* \widehat{N}_Q(\boldsymbol{\theta}) \mathcal{P}_j(\boldsymbol{\theta}), \qquad (10.13)$$

$$\widehat{N}_{*,Q}(\boldsymbol{\theta}) = \sum_{1 \leqslant j,l \leqslant p(\boldsymbol{\theta}): \, j \neq l} \mathcal{P}_j(\boldsymbol{\theta})^* \widehat{N}_Q(\boldsymbol{\theta}) \mathcal{P}_l(\boldsymbol{\theta}).$$

§11. Approximation of the smoothed sandwiched operator $e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})}$

11.1. Approximation of the smoothed sandwiched operator $e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})}$ **in the general case.** We apply theorems of §5 to the operator $\mathcal{A}(\mathbf{k})$. First we apply Theorem 5.7. The constant t^0 is given by (7.12). Taking (7.10) and (7.11) into account, instead of the precise values of the constants C_1 and C_2 which now depend on $\boldsymbol{\theta}$ we take the larger values

$$\begin{aligned}
C_1 &= 2\beta_1 r_0^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}, \\
C_2 &= 2\beta_2 r_0^{-1} \alpha_1^{3/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{3/2} \|g^{-1}\|_{L_\infty}^{1/2} \|f\|_{L_\infty}^3 \|f^{-1}\|_{L_\infty}.
\end{aligned} \qquad (11.1)$$

Applying (5.16) for the operator $\mathcal{A}(\mathbf{k})$ and using (9.2) and (10.4), we obtain

$$\left\| \left(f e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k}, \varepsilon)^{3/2} \widehat{P} \right\|_{L_2(\Omega) \to L_2(\Omega)}$$
$$\leqslant \|f\|_{L_\infty}^2 \|f^{-1}\|_{L_\infty}^2 (C_1 + C_2 |\tau|)\varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |\mathbf{k}| \leqslant t^0.$$
$$(11.2)$$

For $|\mathbf{k}| > t^0$ estimates are trivial. Using the analog of (9.5) for $|\mathbf{k}| > t^0$ and (10.2), we have

$$\| \left(f e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k},\varepsilon)^{1/2} \widehat{P} \|_{L_2(\Omega)\to L_2(\Omega)}$$
$$\leqslant 2\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}(t^0)^{-1}\varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |\mathbf{k}| > t^0.$$
(11.3)

Next, by (9.8),

$$\| \left(f e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k},\varepsilon)^{1/2} (I - \widehat{P}) \|_{L_2(\Omega)\to L_2(\Omega)}$$
$$\leqslant 2\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty} r_0^{-1}\varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad \mathbf{k} \in \widetilde{\Omega}.$$
(11.4)

Finally, relations (11.2)–(11.4) (and expressions for the constants) imply the following result.

Theorem 11.1. *For $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $\mathbf{k} \in \widetilde{\Omega}$ we have*

$$\| \left(f e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k},\varepsilon)^{3/2} \|_{L_2(\Omega)\to L_2(\Omega)}$$
$$\leqslant (\mathcal{C}_1 + \mathcal{C}_2|\tau|)\varepsilon,$$

where

$$\mathcal{C}_1 = \widetilde{\beta}_1 r_0^{-1}\alpha_1^{1/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}\|f\|_{L_\infty}^3\|f^{-1}\|_{L_\infty}^3,$$
$$\mathcal{C}_2 = 2\beta_2 r_0^{-1}\alpha_1^{3/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}\|f\|_{L_\infty}^5\|f^{-1}\|_{L_\infty}^3.$$
(11.5)

11.2. Refinement of approximation of the smoothed sandwiched operator $e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})}$ in the case where $\widehat{N}_Q(\boldsymbol{\theta}) = 0$. Now we assume that $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ and apply Theorem 5.8. By (9.2) and (10.4), we have

$$\| \left(f e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k},\varepsilon)\widehat{P} \|_{L_2(\Omega)\to L_2(\Omega)}$$
$$\leqslant \|f\|_{L_\infty}^2\|f^{-1}\|_{L_\infty}^2(C_1' + C_5|\tau|)\varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |\mathbf{k}| \leqslant t^0.$$

Here $C_1' = \max\{2, C_1\}$, the constant C_1 is defined by (11.1), and the constant C_5 is given by $C_5 = 4\beta_5 r_0^{-2}\alpha_1^2\alpha_0^{-1}\|g\|_{L_\infty}^2\|g^{-1}\|_\infty\|f\|_{L_\infty}^4\|f^{-1}\|_\infty^2$.

Together with (11.3) and (11.4) this implies the following result.

Theorem 11.2. *Let $\widehat{N}_Q(\boldsymbol{\theta})$ be the operator defined by (10.10), (10.11). Suppose that $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $\mathbf{k} \in \widetilde{\Omega}$ we have*

$$\| \left(f e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k},\varepsilon) \|_{L_2(\Omega)\to L_2(\Omega)} \leqslant (\mathcal{C}_3 + \mathcal{C}_4|\tau|)\varepsilon,$$

where

$$\mathcal{C}_3 = \max\{2\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}(\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty} + r_0^{-1}), \mathcal{C}_1\},$$
$$\mathcal{C}_4 = 4\beta_5 r_0^{-2}\alpha_1^2\alpha_0^{-1}\|g\|_{L_\infty}^2\|g^{-1}\|_{L_\infty}\|f\|_{L_\infty}^6\|f^{-1}\|_{L_\infty}^4,$$
(11.6)

and $\mathcal{C}_1$ is defined by (11.5).

Recall that some sufficient conditions ensuring that $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ are given in Proposition 10.1.

11.3. Refinement of approximation of the smoothed sandwiched operator $e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})}$ in the case where $\widehat{N}_{0,Q}(\boldsymbol{\theta}) = 0$. Now we reject the assumption of Theorem 11.2, but instead we assume that $\widehat{N}_{0,Q}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta}$. We may also assume that $\widehat{N}_Q(\boldsymbol{\theta}) = \widehat{N}_{*,Q}(\boldsymbol{\theta}) \neq 0$ for some $\boldsymbol{\theta}$, and then for most points $\boldsymbol{\theta}$ (otherwise, one can apply Theorem 11.2.) As in Subsection 9.5, in order to apply "abstract" Theorem 5.9, we have to impose an additional condition. We use the initial enumeration of the eigenvalues $\gamma_1(\boldsymbol{\theta}), \ldots, \gamma_n(\boldsymbol{\theta})$ of the germ $S(\boldsymbol{\theta})$ (each eigenvalue is repeated corresponding to its multiplicity) and enumerate them in the nondecreasing order:

$$\gamma_1(\boldsymbol{\theta}) \leqslant \gamma_2(\boldsymbol{\theta}) \leqslant \ldots \leqslant \gamma_n(\boldsymbol{\theta}). \tag{11.7}$$

As has been already mentioned, the numbers (11.7) are also the eigenvalues of the generalized spectral problem (10.7). For each $\boldsymbol{\theta}$, let $\mathcal{P}^{(k)}(\boldsymbol{\theta})$ be the "skew" projection (orthogonal with the weight $\overline{Q}$) of $L_2(\Omega; \mathbb{C}^n)$ onto the eigenspace of the problem (10.7) corresponding to the eigenvalue $\gamma_k(\boldsymbol{\theta})$. Clearly, $\mathcal{P}^{(k)}(\boldsymbol{\theta})$ coincides with one of the projections $\mathcal{P}_j(\boldsymbol{\theta})$ introduced in Subsection 10.4 (but the number j may depend on $\boldsymbol{\theta}$).

Condition 11.3. 1°. *The operator $\widehat{N}_{0,Q}(\boldsymbol{\theta})$ defined by (10.13) is equal to zero: $\widehat{N}_{0,Q}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*
2°. *For any pair of indices (k, r), $1 \leqslant k, r \leqslant n$, $k \neq r$, such that $\gamma_k(\boldsymbol{\theta}_0) = \gamma_r(\boldsymbol{\theta}_0)$ for some $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$, we have $(\mathcal{P}^{(k)}(\boldsymbol{\theta}))^*\widehat{N}_Q(\boldsymbol{\theta})\mathcal{P}^{(r)}(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*

Note that, if $\gamma_k(\boldsymbol{\theta}_0) = \gamma_r(\boldsymbol{\theta}_0)$, then the projections $\mathcal{P}^{(k)}(\boldsymbol{\theta}_0)$ and $\mathcal{P}^{(r)}(\boldsymbol{\theta}_0)$ coincide, and the identity $(\mathcal{P}^{(k)}(\boldsymbol{\theta}_0))^*\widehat{N}_Q(\boldsymbol{\theta}_0)\mathcal{P}^{(r)}(\boldsymbol{\theta}_0) = 0$ is valid automatically due to condition 1°. Condition 2° can be reformulated as follows: it is assumed that, for the "blocks" $(\mathcal{P}^{(k)}(\boldsymbol{\theta}))^*\widehat{N}_Q(\boldsymbol{\theta})\mathcal{P}^{(r)}(\boldsymbol{\theta})$ of the operator $\widehat{N}_Q(\boldsymbol{\theta})$ that are not identically zero, the corresponding branches of the eigenvalues $\gamma_k(\boldsymbol{\theta})$ and $\gamma_r(\boldsymbol{\theta})$ do not intersect.
Condition 11.3 is ensured by the following more restrictive condition.

Condition 11.4. 1°. *The operator $\widehat{N}_{0,Q}(\boldsymbol{\theta})$ defined by (10.13) is equal to zero: $\widehat{N}_{0,Q}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*
2°. *Suppose that the number p of different eigenvalues of the generalized spectral problem (10.7) does not depend on $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Denote the different eigenvalues of this problem enumerated in the increasing order by $\gamma_1^\circ(\boldsymbol{\theta}), \ldots, \gamma_p^\circ(\boldsymbol{\theta})$, and assume that their multiplicities $k_1, \ldots, k_p$ do not depend on $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.*

Remark 11.5. Assumption 2° of Condition 11.4 is a fortiori valid if the spectrum of the problem (10.7) is simple for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

So, we assume that Condition 11.3 is satisfied. We have to take care only about the pairs of indices from the set

$$\mathcal{K} := \{(k,r): \ 1 \leqslant k, r \leqslant n, \ k \neq r, \ (\mathcal{P}^{(k)}(\boldsymbol{\theta}))^* \widehat{N}_Q(\boldsymbol{\theta}) \mathcal{P}^{(r)}(\boldsymbol{\theta}) \not\equiv 0\}.$$

Denote (cf. (2.3))

$$c_{kr}^\circ(\boldsymbol{\theta}) := \min\{c_*, n^{-1}|\gamma_k(\boldsymbol{\theta}) - \gamma_r(\boldsymbol{\theta})|\}, \quad (k,r) \in \mathcal{K}.$$

Since the operator-valued function $S(\boldsymbol{\theta})$ is continuous with respect to $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, then $\gamma_j(\boldsymbol{\theta})$ are continuous functions on the sphere $\mathbb{S}^{d-1}$. By Condition 11.3(2°), for $(k,r) \in \mathcal{K}$ we have $|\gamma_k(\boldsymbol{\theta}) - \gamma_r(\boldsymbol{\theta})| > 0$ for all $\boldsymbol{\theta}$, whence

$$c_{kr}^\circ := \min_{\boldsymbol{\theta} \in \mathbb{S}^{d-1}} c_{kr}^\circ(\boldsymbol{\theta}) > 0, \quad (k,r) \in \mathcal{K}.$$

We put

$$c^\circ := \min_{(k,r) \in \mathcal{K}} c_{kr}^\circ. \tag{11.8}$$

Clearly, the number (11.8) plays the role of the number (3.33); it is important that, due to Condition 11.3(2°), we choose c° idependently of $\boldsymbol{\theta}$.

The number t^{00} subject to (3.34) also can be chosen independently of $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Taking (7.10) and (7.11) into account, we put

$$t^{00} = (8\beta_2)^{-1} r_0 \alpha_1^{-3/2} \alpha_0^{1/2} \|g\|_{L_\infty}^{-3/2} \|g^{-1}\|_{L_\infty}^{-1/2} \|f\|_{L_\infty}^{-3} \|f^{-1}\|_{L_\infty}^{-1} c^\circ, \tag{11.9}$$

where c° is given by (11.8). (The condition $t^{00} \leqslant t^0$ is valid automatically, since $c^\circ \leqslant \|S(\boldsymbol{\theta})\| \leqslant \alpha_1 \|g\|_{L_\infty} \|f\|_{L_\infty}^2$.)

Under Condition 11.3, we apply Theorem 5.9 and obtain

$$\left\| \left(f e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k},\varepsilon) \widehat{P} \right\|_{L_2(\Omega) \to L_2(\Omega)}$$
$$\leqslant \|f\|_{L_\infty}^2 \|f^{-1}\|_{L_\infty}^2 (C_9' + C_{10}|\tau|)\varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |\mathbf{k}| \leqslant t^{00}, \tag{11.10}$$

where

$$C_9' = \max\{2, \beta_9 r_0^{-1} \alpha_1^{1/2} \alpha_0^{-1/2} \|g\|_{L_\infty}^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty}$$
$$\times \left(1 + n^2 \alpha_1 \|g\|_{L_\infty} \|f\|_{L_\infty}^2 (c^\circ)^{-1} \right)\},$$
$$C_{10} = \beta_{10} r_0^{-2} \alpha_1^2 \alpha_0^{-1} \|g\|_{L_\infty}^2 \|g^{-1}\|_{L_\infty} \|f\|_{L_\infty}^4 \|f^{-1}\|_{L_\infty}^2$$
$$\times \left(1 + n^2 \alpha_1^2 \|g\|_{L_\infty}^2 \|f\|_{L_\infty}^4 (c^\circ)^{-2} \right). \tag{11.11}$$

Similarly to (11.3), we have

$$\left\| \left(f e^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k},\varepsilon)^{1/2} \widehat{P} \right\|_{L_2(\Omega) \to L_2(\Omega)}$$
$$\leqslant 2\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} (t^{00})^{-1}\varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |\mathbf{k}| > t^{00}. \tag{11.12}$$

Now relations (11.4), (11.10), and (11.12) directly imply the following result.

Theorem 11.6. *Suppose that Condition* 11.3 (*or more restrictive Condition* 11.4) *is satisfied. Then for* $\tau \in \mathbb{R}$, $\varepsilon > 0$, *and* $\mathbf{k} \in \widetilde{\Omega}$ *we have*

$$\left\| \left(f e^{-i\varepsilon^{-2}\tau \mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau \mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k}, \varepsilon) \right\|_{L_2(\Omega) \to L_2(\Omega)}$$
$$\leqslant (\mathcal{C}_5 + \mathcal{C}_6 |\tau|)\varepsilon, \tag{11.13}$$

where $\mathcal{C}_5 = \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \left(\max\{\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} C_9', 2(t^{00})^{-1}\} + 2r_0^{-1} \right),$ $\mathcal{C}_6 = \|f\|_{L_\infty}^2 \|f^{-1}\|_{L_\infty}^2 C_{10},$ *and the constants* C_9', C_{10}, *and* t^{00} *are defined by* (11.11) *and* (11.9).

The assumptions of Theorem 11.6 are a fortiori satisfied in the "real" case, if the spectrum of the problem (10.7) is simple (see Corollary 10.3 and Remark 11.5). We arrive at the following corollary.

Corollary 11.7. *Suppose that the matrices* $b(\boldsymbol{\theta})$, $g(\mathbf{x})$, *and* $Q(\mathbf{x})$ *have real entries. Suppose that the spectrum of the problem* (10.7) *is simple for any* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. *Then estimate* (11.13) *holds for* $\tau \in \mathbb{R}$, $\varepsilon > 0$, *and* $\mathbf{k} \in \widetilde{\Omega}$.

11.4. The sharpness of the result in the general case. Application of Theorem 5.10 allows us to confirm the sharpness of the result of Theorem 11.1 in the general case.

Theorem 11.8. *Let* $\widehat{N}_{0,Q}(\boldsymbol{\theta})$ *be the operator defined by* (10.13). *Suppose that* $\widehat{N}_{0,Q}(\boldsymbol{\theta}_0) \neq 0$ *at some point* $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$. *Let* $0 \neq \tau \in \mathbb{R}$. *Then for any* $1 \leqslant s < 3$ *it is impossible that the estimate*

$$\left\| \left(f e^{-i\varepsilon^{-2}\tau \mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau \mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k}, \varepsilon)^{s/2} \right\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant \mathcal{C}(\tau)\varepsilon \tag{11.14}$$

holds for almost every $\mathbf{k} = t\boldsymbol{\theta} \in \widetilde{\Omega}$ *and sufficiently small* $\varepsilon > 0$.

For the proof we need the following lemma which can be easily checked by analogy with the proof of Lemma 9.10.

Lemma 11.9. *Let* δ *and* t^0 *be given by* (7.10) *and* (7.12), *respectively. Let* $F(\mathbf{k}) = F(t, \boldsymbol{\theta})$ *be the spectral projection of the operator* $\mathcal{A}(\mathbf{k})$ *for the interval* $[0, \delta]$. *Then for* $|\mathbf{k}| \leqslant t^0$ *and* $|\mathbf{k}_0| \leqslant t^0$ *we have*

$$\|F(\mathbf{k}) - F(\mathbf{k}_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant C'|\mathbf{k} - \mathbf{k}_0|,$$

$$\|\mathcal{A}(\mathbf{k})F(\mathbf{k}) - \mathcal{A}(\mathbf{k}_0)F(\mathbf{k}_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant C''|\mathbf{k} - \mathbf{k}_0|,$$

$$\|e^{-i\tau \mathcal{A}(\mathbf{k})}F(\mathbf{k}) - e^{-i\tau \mathcal{A}(\mathbf{k}_0)}F(\mathbf{k}_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant (2C' + C''|\tau|)|\mathbf{k} - \mathbf{k}_0|.$$

Proof of Theorem 11.8. We prove by contradiction. Let us fix $\tau \neq 0$. Suppose that for some $1 \leqslant s < 3$ there exists a constant $\mathcal{C}(\tau) > 0$ such that estimate (11.14) holds for almost every $\mathbf{k} \in \widetilde{\Omega}$ and sufficiently small $\varepsilon > 0$. By (11.4) and (9.2), it follows that there exists a constant $\widetilde{\mathcal{C}}(\tau) > 0$ such that

$$\left\| \left(f e^{-i\varepsilon^{-2}\tau \mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau \mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \widehat{P} \right\| \varepsilon^s (|\mathbf{k}|^2 + \varepsilon^2)^{-s/2} \leqslant \widetilde{\mathcal{C}}(\tau)\varepsilon \tag{11.15}$$

for almost every $\mathbf{k} \in \widetilde{\Omega}$ and sufficiently small ε.

By (5.2), we have $f^{-1}\widehat{P} = Pf^*\overline{Q}$, where P is the orthogonal projection of $L_2(\Omega;\mathbb{C}^n)$ onto the subspace $\mathfrak{N}$ (see (7.3)). Then the operator under the norm sign in (11.15) can be written as $fe^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})}Pf^*\overline{Q} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})}f_0^{-1}\widehat{P}$.

Now, let $|\mathbf{k}| \leqslant t^0$. By (1.13), $\|F(\mathbf{k}) - P\| \leqslant C_1|\mathbf{k}|$ for $|\mathbf{k}| \leqslant t^0$. Together with (11.15) this implies that there exists a constant $\check{\mathcal{C}}(\tau)$ such that

$$\|fe^{-i\varepsilon^{-2}\tau\mathcal{A}(\mathbf{k})}F(\mathbf{k})f^*\overline{Q} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(\mathbf{k})}f_0^{-1}\widehat{P}\|\varepsilon^s(|\mathbf{k}|^2 + \varepsilon^2)^{-s/2} \leqslant \check{\mathcal{C}}(\tau)\varepsilon \tag{11.16}$$

for almost every $\mathbf{k}$ in the ball $|\mathbf{k}| \leqslant t^0$ and sufficiently small ε.

Observe that $\widehat{P}$ is the spectral projection of the operator $\mathcal{A}^0(\mathbf{k})$ for the interval $[0, \delta]$. Therefore, Lemma 11.9 (applied to $\mathcal{A}(\mathbf{k})$ and $\mathcal{A}^0(\mathbf{k})$) implies that for fixed τ and ε the operator under the norm sign in (11.16) is continuous with respect to $\mathbf{k}$ in the ball $|\mathbf{k}| \leqslant t^0$. Hence, estimate (11.16) holds for all $\mathbf{k}$ in this ball. In particular, it is satisfied at the point $\mathbf{k} = t\boldsymbol{\theta}_0$ if $t \leqslant t^0$. Applying once more the inequality $\|F(\mathbf{k}) - P\| \leqslant C_1|\mathbf{k}|$ and the identity $Pf^*\overline{Q} = f^{-1}\widehat{P}$, we see that the inequality

$$\left\| \left(fe^{-i\varepsilon^{-2}\tau\mathcal{A}(t\boldsymbol{\theta}_0)}f^{-1} - f_0 e^{-i\varepsilon^{-2}\tau\mathcal{A}^0(t\boldsymbol{\theta}_0)}f_0^{-1} \right) \widehat{P}\|\varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leqslant \check{\mathcal{C}}'(\tau)\varepsilon \tag{11.17}$$

holds for all $t \leqslant t^0$ and sufficiently small ε.

In abstract terms, estimate (11.17) corresponds to the inequality (5.17). Since we assume that $\widehat{N}_{0,Q}(\boldsymbol{\theta}_0) \neq 0$, applying Theorem 5.10, we arrive at a contradiction. $\bullet$

§12. Approximation of the smoothed operator $e^{-i\varepsilon^{-2}\tau\mathcal{A}}$

12.1. Approximation of the smoothed operator $e^{-i\varepsilon^{-2}\tau\widehat{\mathcal{A}}}$. In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the operator $\widehat{\mathcal{A}} = b(\mathbf{D})^*g(\mathbf{x})b(\mathbf{D})$ (see (8.1)). Let $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^*g^0 b(\mathbf{D})$ be the effective operator (see (8.11)). Recall the notation $\mathcal{H}_0 = -\Delta$ and put

$$\mathcal{R}(\varepsilon) := \varepsilon^2(\mathcal{H}_0 + \varepsilon^2 I)^{-1}. \tag{12.1}$$

Expansion (7.7) for $\widehat{\mathcal{A}}$ yields

$$e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}} = \mathcal{U}^{-1}\left(\int_{\widetilde{\Omega}} \oplus e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}(\mathbf{k})}\, d\mathbf{k} \right)\mathcal{U}.$$

The operator $e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0}$ admits a similar representation. The operator (12.1) expands in the direct integral of the operators (9.1):

$$\mathcal{R}(\varepsilon) = \mathcal{U}^{-1}\left(\int_{\widetilde{\Omega}} \oplus\mathcal{R}(\mathbf{k},\varepsilon)\, d\mathbf{k} \right)\mathcal{U}. \tag{12.2}$$

It follows that the operator $\left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0} \right)\mathcal{R}(\varepsilon)^{s/2}$ expands in the direct integral of the operators $\left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0(\mathbf{k})} \right)\mathcal{R}(\mathbf{k},\varepsilon)^{s/2}$.

Hence,

$$
\begin{aligned}
&\| \left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0} \right) \mathcal{R}(\varepsilon)^{s/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \\
&= \operatorname*{ess\,sup}_{\mathbf{k}\in\widetilde{\Omega}} \| \left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}(\mathbf{k})} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0(\mathbf{k})} \right) \mathcal{R}(\mathbf{k},\varepsilon)^{s/2} \|_{L_2(\Omega) \to L_2(\Omega)}.
\end{aligned}
\tag{12.3}
$$

Therefore, Theorem 9.1 directly implies the following statement.

Theorem 12.1. *Let $\widehat{\mathcal{A}}$ be the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by $\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$, where $g(\mathbf{x})$ and $b(\mathbf{D})$ satisfy the assumptions of Subsection 6.1. Let $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ be the effective operator, where g^0 is given by (8.9). Let $\mathcal{R}(\varepsilon)$ be defined by (12.1). Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$
\| \left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0} \right) \mathcal{R}(\varepsilon)^{3/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\widehat{\mathcal{C}}_1 + \widehat{\mathcal{C}}_2|\tau|)\varepsilon.
$$

The constants $\widehat{\mathcal{C}}_1$ and $\widehat{\mathcal{C}}_2$ are defined by (9.10) and depend only on r_0, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

Similarly, Theorem 9.2 implies the following result.

Theorem 12.2. *Suppose that the assumptions of Theorem 12.1 are satisfied. Let $\widehat{N}(\boldsymbol{\theta})$ be the operator defined by (8.18), (8.19). Suppose that $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$
\| \left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0} \right) \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\widehat{\mathcal{C}}_3 + \widehat{\mathcal{C}}_4|\tau|)\varepsilon.
$$

The constants $\widehat{\mathcal{C}}_3$ and $\widehat{\mathcal{C}}_4$ are defined in Theorem 9.2 and depend only on r_0, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

Recall that some sufficient conditions ensuring that the assumptions of Theorem 12.2 are satisfied are given in Proposition 8.4.

Finally, applying Theorem 9.7 and using the direct integral expansion, we obtain the following statement.

Theorem 12.3. *Suppose that the assumptions of Theorem 12.1 are satisfied. Suppose also that Condition 9.3 (or more restrictive Condition 9.4) is satisfied. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$
\| \left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0} \right) \mathcal{R}(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\widehat{\mathcal{C}}_5 + \widehat{\mathcal{C}}_6|\tau|)\varepsilon.
$$

The constants $\widehat{\mathcal{C}}_5$ and $\widehat{\mathcal{C}}_6$ are defined in Theorem 9.7 and depend only on r_0, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and also on the number $\widehat{c}^\circ$ defined by (9.11).

Recall that some sufficient conditions ensuring that the assumptions of Theorem 12.3 are satisfied are given in Corollary 9.8.

Applying Theorem 9.9, we confirm the sharpness of the result of Theorem 12.1.

Theorem 12.4. *Suppose that the assumptions of Theorem 12.1 are satisfied. Let $\widehat{N}_0(\boldsymbol{\theta})$ be the operator defined by (8.21). Suppose that $\widehat{N}_0(\boldsymbol{\theta}_0) \neq 0$ for*

some $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$. Let $0 \neq \tau \in \mathbb{R}$. Then for any $1 \leqslant s < 3$ it is impossible that the estimate

$$\left\| \left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0} \right) \mathcal{R}(\varepsilon)^{s/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathcal{C}(\tau)\varepsilon \tag{12.4}$$

holds for all sufficiently small $\varepsilon > 0$.

Proof. We prove by contradiction. Let us fix $\tau \neq 0$. Suppose that for some $1 \leqslant s < 3$ there exists a constant $\mathcal{C}(\tau) > 0$ such that (12.4) holds for all sufficiently small ε. By (12.3), this means that for almost every $\mathbf{k} \in \widetilde{\Omega}$ and sufficiently small ε estimate (9.17) holds. But this contradicts to the statement of Theorem 9.9. $\bullet$

12.2.   Approximation of the smoothed sandwiched operator $e^{-i\varepsilon^{-2}\tau\mathcal{A}}$.   In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, consider the operator $\mathcal{A} = f(\mathbf{x})^* \widehat{\mathcal{A}} f(\mathbf{x}) = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x})$ (see (6.4)). Let $\widehat{\mathcal{A}}^0$ be the operator (8.11), and let f_0 be the matrix (10.1). Let $\mathcal{A}^0 = f_0 \widehat{\mathcal{A}}^0 f_0 = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D}) f_0$ (see (10.3)).

Similarly to (12.3), using (7.7) and (12.2), we have

$$\left\| \left(f e^{-i\tau\varepsilon^{-2}\mathcal{A}} f^{-1} - f_0 e^{-i\tau\varepsilon^{-2}\mathcal{A}^0} f_0^{-1} \right) \mathcal{R}(\varepsilon)^{s/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}$$

$$= \underset{\mathbf{k} \in \widetilde{\Omega}}{\text{ess-sup}} \left\| \left(f e^{-i\tau\varepsilon^{-2}\mathcal{A}(\mathbf{k})} f^{-1} - f_0 e^{-i\tau\varepsilon^{-2}\mathcal{A}^0(\mathbf{k})} f_0^{-1} \right) \mathcal{R}(\mathbf{k}, \varepsilon)^{s/2} \right\|_{L_2(\Omega) \to L_2(\Omega)}.$$

$$\tag{12.5}$$

Theorem 11.1 together with (12.5) implies the following result.

Theorem 12.5. *Let $\mathcal{A}$ be the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by $\mathcal{A} = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x})$, where $g(\mathbf{x})$, $f(\mathbf{x})$ and $b(\mathbf{D})$ satisfy the assumptions of Subsection 6.1. Let $\mathcal{A}^0 = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D}) f_0$, where g^0 is the effective matrix (8.9) and $f_0 = (\underline{f f^*})^{1/2}$. Let $\mathcal{R}(\varepsilon)$ be defined by (12.1). Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\left\| \left(f e^{-i\tau\varepsilon^{-2}\mathcal{A}} f^{-1} - f_0 e^{-i\tau\varepsilon^{-2}\mathcal{A}^0} f_0^{-1} \right) \mathcal{R}(\varepsilon)^{3/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\mathcal{C}_1 + \mathcal{C}_2|\tau|)\varepsilon.$$

The constants $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined by (11.5) and depend only on r_0, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, and $\|f^{-1}\|_{L_\infty}$.

Similarly, Theorem 11.2 leads to the following statement.

Theorem 12.6. *Suppose that the assumptions of Theorem 12.5 are satisfied. Let $\widehat{N}_Q(\boldsymbol{\theta})$ be the operator defined by (10.10), (10.11). Suppose that $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\left\| \left(f e^{-i\tau\varepsilon^{-2}\mathcal{A}} f^{-1} - f_0 e^{-i\tau\varepsilon^{-2}\mathcal{A}^0} f_0^{-1} \right) \mathcal{R}(\varepsilon) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\mathcal{C}_3 + \mathcal{C}_4|\tau|)\varepsilon.$$

The constants $\mathcal{C}_3$ and $\mathcal{C}_4$ are given by (11.6) and depend only on r_0, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, and $\|f^{-1}\|_{L_\infty}$.

Recall that some sufficient conditions ensuring that the assumptions of Theorem 12.6 are satisfiied are given in Proposition 10.1.

Finally, from Theorem 11.6 and the direct integral expansion we deduce the following result.

Theorem 12.7. *Suppose that the assumptions of Theorem* 12.5 *are satisfied. Suppose also that Condition* 11.3 (*or more restrictive Condition* 11.4) *is satisfied. Then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\left\| \left(f e^{-i\tau\varepsilon^{-2}\mathcal{A}} f^{-1} - f_0 e^{-i\tau\varepsilon^{-2}\mathcal{A}^0} f_0^{-1} \right) \mathcal{R}(\varepsilon) \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\mathcal{C}_5 + \mathcal{C}_6 |\tau|)\varepsilon.$$

The constants $\mathcal{C}_5$ *and* $\mathcal{C}_6$ *are defined in Theorem* 11.6 *and depend only on* r_0, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, *and also on the number* c° *defined by* (11.8).

Recall that some sufficient conditions ensuring that the assumptions of Theorem 12.7 are satisfied are given in Corollary 11.7.

By analogy with the proof of Theorem 12.4, we deduce the following result from Theorem 11.8; this confirms that the result of Theorem 12.5 is sharp.

Theorem 12.8. *Suppose that the assumptions of Theorem* 12.5 *are satisfied. Let* $\widehat{N}_{0,Q}(\boldsymbol{\theta})$ *be the operator defined by* (10.13). *Suppose that* $\widehat{N}_{0,Q}(\boldsymbol{\theta}_0) \neq 0$ *for some* $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$. *Let* $0 \neq \tau \in \mathbb{R}$. *Then for any* $1 \leqslant s < 3$ *it is impossible that the estimate*

$$\left\| \left(f e^{-i\tau\varepsilon^{-2}\mathcal{A}} f^{-1} - f_0 e^{-i\tau\varepsilon^{-2}\mathcal{A}^0} f_0^{-1} \right) \mathcal{R}(\varepsilon)^{s/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathcal{C}(\tau)\varepsilon \quad (12.6)$$

holds for all sufficiently small $\varepsilon > 0$.

<div align="center">

CHAPTER 3. HOMOGENIZATION PROBLEMS FOR NONSTATIONARY
SCHRÖDINGER TYPE EQUATION

§13. HOMOGENIZATION OF THE OPERATOR EXPONENTIAL $e^{-i\tau\mathcal{A}_\varepsilon}$

</div>

13.1. The operators $\widehat{\mathcal{A}}_\varepsilon$ and $\mathcal{A}_\varepsilon$. Statement of the problem. If $\phi(\mathbf{x})$ is a Γ-periodic function in $\mathbb{R}^d$, we agree to denote $\phi^\varepsilon(\mathbf{x}) := \phi(\varepsilon^{-1}\mathbf{x})$, $\varepsilon > 0$. *Our main objects* are the operators $\widehat{\mathcal{A}}_\varepsilon$ and $\mathcal{A}_\varepsilon$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and formally given by

$$\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}), \quad \mathcal{A}_\varepsilon = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}). \quad (13.1)$$

The precise definitions are given in terms of the corresponding quadratic forms (cf. Subsection 6.1). The coefficients of the operators (13.1) oscillate rapidly as $\varepsilon \to 0$.

Our goal is to find approximations for small ε for the operator exponential $e^{-i\tau\widehat{\mathcal{A}}_\varepsilon}$ and for the sandwiched exponential $f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon} (f^\varepsilon)^{-1}$ and to apply the results to homogenization of the Cauchy problem for the Schrödinger type equation.

13.2. The scaling transformation. Let T_ε be the unitary scaling transformation in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ defined by $(T_\varepsilon \mathbf{u})(\mathbf{x}) = \varepsilon^{d/2}\mathbf{u}(\varepsilon\mathbf{x})$, $\varepsilon > 0$. Then $\mathcal{A}_\varepsilon = \varepsilon^{-2} T_\varepsilon^* \mathcal{A} T_\varepsilon$. Hence,

$$e^{-i\tau\mathcal{A}_\varepsilon} = T_\varepsilon^* e^{-i\tau\varepsilon^{-2}\mathcal{A}} T_\varepsilon. \quad (13.2)$$

The operator $\widehat{\mathcal{A}}_\varepsilon$ satisfies similar relations.

Applying the scaling transformation to the resolvent of the operator $\mathcal{H}_0 = -\Delta$, we obtain

$$(\mathcal{H}_0 + I)^{-1} = \varepsilon^2 T_\varepsilon^*(\mathcal{H}_0 + \varepsilon^2 I)^{-1}T_\varepsilon = T_\varepsilon^* \mathcal{R}(\varepsilon)T_\varepsilon. \tag{13.3}$$

Here we have used the notation (12.1).

Finally, if $\phi(\mathbf{x})$ is a Γ-periodic function, then, under the scaling transformation, the operator $[\phi^\varepsilon]$ of multiplication by the function $\phi^\varepsilon(\mathbf{x})$ turns into the operator $[\phi]$ of multiplication by $\phi(\mathbf{x})$:

$$[\phi^\varepsilon] = T_\varepsilon^*[\phi]T_\varepsilon. \tag{13.4}$$

13.3. Approximation of the operator $e^{-i\tau\widehat{\mathcal{A}}_\varepsilon}$. We start with the simpler operator $\widehat{\mathcal{A}}_\varepsilon$. Let $\widehat{\mathcal{A}}^0$ be the effective operator (8.11). Using relations of the form (13.2) (for the operators $\widehat{\mathcal{A}}_\varepsilon$ and $\widehat{\mathcal{A}}^0$) and (13.3), we obtain

$$\begin{aligned}
&\left(e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\right)(\mathcal{H}_0 + I)^{-s/2} \\
&= T_\varepsilon^*\left(e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}} - e^{-i\tau\varepsilon^{-2}\widehat{\mathcal{A}}^0}\right)\mathcal{R}(\varepsilon)^{s/2}T_\varepsilon, \quad \varepsilon > 0.
\end{aligned} \tag{13.5}$$

Since T_ε is unitary, combining this with Theorem 12.1, we deduce the following result (which has been proved before in [BSu5, Theorem 12.1]).

Theorem 13.1. *Let $\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})$, where $g(\mathbf{x})$ and $b(\mathbf{D})$ satisfy the assumptions of Subsection 6.1. Let $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ be the effective operator, where g^0 is given by (8.9). Let $\mathcal{H}_0 = -\Delta$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|\left(e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\right)(\mathcal{H}_0 + I)^{-3/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant (\widehat{\mathcal{C}}_1 + \widehat{\mathcal{C}}_2|\tau|)\varepsilon. \tag{13.6}$$

The constants $\widehat{\mathcal{C}}_1$ and $\widehat{\mathcal{C}}_2$ are given by (9.10) and depend only on r_0, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

Obviously,

$$\|e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant 2, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \tag{13.7}$$

Interpolating between (13.7) and (13.6), for $0 \leqslant s \leqslant 3$ we obtain

$$\begin{aligned}
&\|\left(e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\right)(\mathcal{H}_0 + I)^{-s/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\
&\leqslant 2^{1-s/3}(\widehat{\mathcal{C}}_1 + \widehat{\mathcal{C}}_2|\tau|)^{s/3}\varepsilon^{s/3}, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0.
\end{aligned} \tag{13.8}$$

The operator $(\mathcal{H}_0 + I)^{s/2}$ is an isometric isomorphism of the Sobolev space $H^s(\mathbb{R}^d; \mathbb{C}^n)$ onto $L_2(\mathbb{R}^d; \mathbb{C}^n)$. Therefore, (13.8) is equivalent to

$$\|e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant 2^{1-s/3}(\widehat{\mathcal{C}}_1 + \widehat{\mathcal{C}}_2|\tau|)^{s/3}\varepsilon^{s/3}, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \tag{13.9}$$

In particular, for $0 < \varepsilon \leqslant 1$ estimate (13.9) allows us to consider large values of time τ, namely, we can consider $|\tau| = O(\varepsilon^{-\alpha})$ for $0 < \alpha < 1$. We arrive at the following theorem (which has been proved before in [BSu5, Theorem 12.2]).

Theorem 13.2. *Suppose that the assumptions of Theorem* 13.1 *are satisfied. Then for* $0 \leqslant s \leqslant 3$, $\tau \in \mathbb{R}$, *and* $\varepsilon > 0$ *we have*

$$\|e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \widehat{\mathfrak{C}}_1(s;\tau)\varepsilon^{s/3}, \qquad (13.10)$$

where

$$\widehat{\mathfrak{C}}_1(s;\tau) = 2^{1-s/3}(\widehat{\mathcal{C}}_1 + \widehat{\mathcal{C}}_2|\tau|)^{s/3}. \qquad (13.11)$$

In particular, for $0 < \varepsilon \leqslant 1$ *and* $|\tau| = \varepsilon^{-\alpha}$, $0 < \alpha < 1$, *we have*

$$\|e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \widehat{\mathfrak{C}}_1(s;1)\varepsilon^{s(1-\alpha)/3}, \quad 0 < \varepsilon \leqslant 1, \quad |\tau| = \varepsilon^{-\alpha}.$$
$$(13.12)$$

13.4. Refinement of approximation of the operator $e^{-i\tau\widehat{\mathcal{A}}_\varepsilon}$ **under the additional assumptions.** Similarly, from (13.5) and Theorem 12.2 we deduce the following result.

Theorem 13.3. *Suppose that the assumptions of Theorem* 13.1 *are satisfied. Let* $\widehat{N}(\boldsymbol{\theta})$ *be the operator defined by* (8.18), (8.19). *Suppose that* $\widehat{N}(\boldsymbol{\theta}) = 0$ *for all* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. *Then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\left\| \left(e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0} \right) (\mathcal{H}_0 + I)^{-1} \right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant (\widehat{\mathcal{C}}_3 + \widehat{\mathcal{C}}_4|\tau|)\varepsilon. \qquad (13.13)$$

The constants $\widehat{\mathcal{C}}_3$ *and* $\widehat{\mathcal{C}}_4$ *are defined in Theorem* 9.2 *and depend only on* r_0, α_0, α_1, $\|g\|_{L_\infty}$, *and* $\|g^{-1}\|_{L_\infty}$.

Interpolating between (13.7) and (13.13), we obtain the following statement.

Theorem 13.4. *Suppose that the assumptions of Theorem* 13.3 *are satisfied. Then for* $0 \leqslant s \leqslant 2$, $\tau \in \mathbb{R}$, *and* $\varepsilon > 0$ *we have*

$$\|e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \widehat{\mathfrak{C}}_2(s;\tau)\varepsilon^{s/2},$$

where

$$\widehat{\mathfrak{C}}_2(s;\tau) = 2^{1-s/2}(\widehat{\mathcal{C}}_3 + \widehat{\mathcal{C}}_4|\tau|)^{s/2}. \qquad (13.14)$$

In particular, for $0 < \varepsilon \leqslant 1$ *and* $|\tau| = \varepsilon^{-\alpha}$, $0 < \alpha < 1$, *we have*

$$\|e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \widehat{\mathfrak{C}}_2(s;1)\varepsilon^{s(1-\alpha)/2}, \quad 0 < \varepsilon \leqslant 1, \quad |\tau| = \varepsilon^{-\alpha}.$$

Theorem 13.4 and Proposition 8.4 imply the following statement.

Corollary 13.5. *Suppose that at least one of the following conditions is fulfilled*:
1°. *The operator* $\widehat{\mathcal{A}}_\varepsilon$ *has the form* $\widehat{\mathcal{A}}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$, *where* $g(\mathbf{x})$ *is a symmetric matrix with real entries.*
2°. *Relations* (8.14) *are satisfied, i. e.,* $g^0 = \overline{g}$.
3°. *Relations* (8.15) *are satisfied, i. e.,* $g^0 = \underline{g}$. *(In particular, this is true if* $m = n$.)
Then the statements of Theorem 13.4 *are valid.*

Finally, Theorem 12.3 and (13.5) lead to the following result.

Theorem 13.6. *Suppose that the assumptions of Theorem* 13.1 *are satisfied. Suppose that Condition* 9.3 (*or more restrictive Condition* 9.4) *is satisfied. Then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\| \left(e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0} \right) (\mathcal{H}_0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant (\widehat{\mathcal{C}}_5 + \widehat{\mathcal{C}}_6|\tau|)\varepsilon. \quad (13.15)$$

The constants $\widehat{\mathcal{C}}_5$ *and* $\widehat{\mathcal{C}}_6$ *are defined in Theorem* 9.7 *and depend only on* r_0, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and on the number* $\widehat{c}^\circ$ *given by* (9.11).

Interpolating between (13.7) and (13.15), we obtain the following theorem.

Theorem 13.7. *Suppose that the assumptions of Theorem* 13.6 *are satisfied. Then for* $0 \leqslant s \leqslant 2$, $\tau \in \mathbb{R}$, *and* $\varepsilon > 0$ *we have*

$$\|e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \widehat{\mathfrak{C}}_3(s;\tau)\varepsilon^{s/2},$$

where

$$\widehat{\mathfrak{C}}_3(s;\tau) = 2^{1-s/2}(\widehat{\mathcal{C}}_5 + \widehat{\mathcal{C}}_6|\tau|)^{s/2}. \quad (13.16)$$

In particular, for $0 < \varepsilon \leqslant 1$ *and* $|\tau| = \varepsilon^{-\alpha}$, $0 < \alpha < 1$, *we have*

$$\|e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0}\|_{H^s(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \widehat{\mathfrak{C}}_3(s;1)\varepsilon^{s(1-\alpha)/2}, \quad 0 < \varepsilon \leqslant 1, \quad |\tau| = \varepsilon^{-\alpha}.$$

Theorem 13.7 and Corollary 9.8 imply the following statement.

Corollary 13.8. *Suppose that the matrices* $b(\boldsymbol{\theta})$ *and* $g(\mathbf{x})$ *have real entries. Suppose that the spectrum of the germ* $\widehat{S}(\boldsymbol{\theta})$ *is simple for all* $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. *Then the statements of Theorem* 13.7 *are valid.*

13.5. The sharpness of the result. Applying Theorem 12.4, we confirm the sharpness of the result of Theorem 13.1 in the general case.

Theorem 13.9. *Suppose that the assumptions of Theorem* 13.1 *are satisfied. Let* $\widehat{N}_0(\boldsymbol{\theta})$ *be the operator defined by* (8.21). *Suppose that* $\widehat{N}_0(\boldsymbol{\theta}_0) \neq 0$ *for some* $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$. *Let* $0 \neq \tau \in \mathbb{R}$. *Then for any* $1 \leqslant s < 3$ *it is impossible that the estimate*

$$\| \left(e^{-i\tau\widehat{\mathcal{A}}_\varepsilon} - e^{-i\tau\widehat{\mathcal{A}}^0} \right) (\mathcal{H}_0 + I)^{-s/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathcal{C}(\tau)\varepsilon \quad (13.17)$$

holds for all sufficiently small $\varepsilon > 0$.

Proof. We prove by contradiction. Let us fix $\tau \neq 0$. Suppose that for some $1 \leqslant s < 3$ there exists a constant $\mathcal{C}(\tau)$ such that estimate (13.17) holds for all sufficiently small ε. Then, by (13.5), estimate (12.4) is also satisfied. But this contradicts to the statement of Theorem 12.4. ●

13.6. Approximation of the operator $f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon}(f^\varepsilon)^{-1}$**.** Now we proceed to the study of the operator $\mathcal{A}_\varepsilon$ (see (13.1)). Let $\mathcal{A}^0$ be defined by (10.3). Using (13.2) for the operators $\mathcal{A}_\varepsilon$ and $\mathcal{A}^0$ and taking (13.3) and (13.4) into account, we obtain the following analog of identity (13.5):

$$\begin{aligned}
&\left(f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon}(f^\varepsilon)^{-1} - f_0 e^{-i\tau\mathcal{A}^0}f_0^{-1} \right) (\mathcal{H}_0 + I)^{-s/2} \\
&= T_\varepsilon^* \left(f e^{-i\tau\varepsilon^{-2}\mathcal{A}}f^{-1} - f_0 e^{-i\tau\varepsilon^{-2}\mathcal{A}^0}f_0^{-1} \right) \mathcal{R}(\varepsilon)^{s/2}T_\varepsilon, \quad \varepsilon > 0.
\end{aligned} \quad (13.18)$$

Since T_ε is unitary, combining this with Theorem 12.5, we obtain the following result (which has been proved before in [BSu5, Theorem 12.3]).

Theorem 13.10. *Let $\mathcal{A}_\varepsilon = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon b(\mathbf{D}) f^\varepsilon(\mathbf{x})$, where $g(\mathbf{x})$, $f(\mathbf{x})$ and $b(\mathbf{D})$ satisfy the assumptions of Subsection 6.1. Let $\mathcal{A}^0 = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D}) f_0$, where g^0 is the effective matrix (8.9) and $f_0 = (\underline{f f^*})^{1/2}$. Let $\mathcal{H}_0 = -\Delta$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\| \left(f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon} (f^\varepsilon)^{-1} - f_0 e^{-i\tau\mathcal{A}^0} f_0^{-1} \right) (\mathcal{H}_0+I)^{-3/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\mathcal{C}_1 + \mathcal{C}_2 |\tau|)\varepsilon.$$
$$(13.19)$$

The constants $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined by (11.5) and depend only on r_0, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, and $\|f^{-1}\|_{L_\infty}$.

Obviously, by (10.2),

$$\| f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon} (f^\varepsilon)^{-1} - f_0 e^{-i\tau\mathcal{A}^0} f_0^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant 2\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty},$$
$$\tau \in \mathbb{R}, \quad \varepsilon > 0.$$
$$(13.20)$$

Interpolating between (13.20) and (13.19), we arrive at the following result which has been proved before in [BSu5, Theorem 12.4].

Theorem 13.11. *Suppose that the assumptions of Theorem 13.10 are satisfied. Then for $0 \leqslant s \leqslant 3$, $\tau \in \mathbb{R}$, and $\varepsilon > 0$ we have*

$$\| f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon} (f^\varepsilon)^{-1} - f_0 e^{-i\tau\mathcal{A}^0} f_0^{-1} \|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_1(s;\tau)\varepsilon^{s/3},$$

where

$$\mathfrak{C}_1(s;\tau) = (2\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty})^{1-s/3} (\mathcal{C}_1 + \mathcal{C}_2|\tau|)^{s/3}. \tag{13.21}$$

In particular, for $0 < \varepsilon \leqslant 1$ and $|\tau| = \varepsilon^{-\alpha}$, $0 < \alpha < 1$, we have

$$\| f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon} (f^\varepsilon)^{-1} - f_0 e^{-i\tau\mathcal{A}^0} f_0^{-1} \|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_1(s;1)\varepsilon^{s(1-\alpha)/3},$$
$$0 < \varepsilon \leqslant 1, \quad |\tau| = \varepsilon^{-\alpha}.$$

13.7. Refinement of approximation of the operator $f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon} (f^\varepsilon)^{-1}$ under the additional assumptions. Using (13.18) and Theorem 12.6, we deduce the following result.

Theorem 13.12. *Suppose that the assumptions of Theorem 13.10 are satisfied. Let $\widehat{N}_Q(\boldsymbol{\theta})$ be the operator defined by (10.10), (10.11). Suppose that $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\| \left(f^\varepsilon e^{-i\tau\mathcal{A}_\varepsilon} (f^\varepsilon)^{-1} - f_0 e^{-i\tau\mathcal{A}^0} f_0^{-1} \right) (\mathcal{H}_0+I)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\mathcal{C}_3 + \mathcal{C}_4 |\tau|)\varepsilon.$$
$$(13.22)$$

The constants $\mathcal{C}_3$ and $\mathcal{C}_4$ are defined by (11.6) and depend only on r_0, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, and $\|f^{-1}\|_{L_\infty}$.

Interpolating between (13.20) and (13.22), we obtain the following statement.

Theorem 13.13. *Suppose that the assumptions of Theorem 13.12 are satisfied. Then for $0 \leqslant s \leqslant 2$, $\tau \in \mathbb{R}$, and $\varepsilon > 0$ we have*

$$\|f^{\varepsilon} e^{-i\tau \mathcal{A}_{\varepsilon}}(f^{\varepsilon})^{-1} - f_0 e^{-i\tau \mathcal{A}^0} f_0^{-1}\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_2(s; \tau) \varepsilon^{s/2},$$

where

$$\mathfrak{C}_2(s; \tau) = (2\|f\|_{L_{\infty}} \|f^{-1}\|_{L_{\infty}})^{1-s/2} (\mathcal{C}_3 + \mathcal{C}_4 |\tau|)^{s/2}. \qquad (13.23)$$

In particular, for $0 < \varepsilon \leqslant 1$ and $|\tau| = \varepsilon^{-\alpha}$, $0 < \alpha < 1$, we have

$$\|f^{\varepsilon} e^{-i\tau \mathcal{A}_{\varepsilon}}(f^{\varepsilon})^{-1} - f_0 e^{-i\tau \mathcal{A}^0} f_0^{-1}\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_2(s; 1) \varepsilon^{s(1-\alpha)/2},$$
$$0 < \varepsilon \leqslant 1, \quad |\tau| = \varepsilon^{-\alpha}.$$

From Theorem 13.13 and Proposition 10.1 we deduce the following corollary.

Corollary 13.14. *Suppose that at least one of the following conditions is fulfilled:*
1°. The operator $\mathcal{A}_{\varepsilon}$ has the form $\widehat{\mathcal{A}}_{\varepsilon} = (f^{\varepsilon})^* \mathbf{D}^* g^{\varepsilon}(\mathbf{x}) \mathbf{D} f^{\varepsilon}(\mathbf{x})$, where $g(\mathbf{x})$ is a symmetric matrix with real entries.
2°. Relations (8.14) are satisfied, i. e. $g^0 = \overline{g}$.
Then the statements of Theorem 13.13 are valid.

Finally, Theorem 12.7 and (13.18) imply the following result.

Theorem 13.15. *Suppose that the assumptions of Theorem 13.10 are satisfied. Suppose that Condition 11.3 (or more restrictive Condition 11.4) is satisfied. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\left\| \left(f^{\varepsilon} e^{-i\tau \mathcal{A}_{\varepsilon}}(f^{\varepsilon})^{-1} - f_0 e^{-i\tau \mathcal{A}^0} f_0^{-1} \right) (\mathcal{H}_0 + I)^{-1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant (\mathcal{C}_5 + \mathcal{C}_6 |\tau|) \varepsilon.$$
$$(13.24)$$

The constants $\mathcal{C}_5$ and $\mathcal{C}_6$ are defined in Theorem 11.6 and depend only on r_0, α_0, α_1, $\|g\|_{L_{\infty}}$, $\|g^{-1}\|_{L_{\infty}}$, $\|f\|_{L_{\infty}}$, $\|f^{-1}\|_{L_{\infty}}$, and also on the number c° defined by (11.8).

Interpolating between (13.20) and (13.24), we obtain the following theorem.

Theorem 13.16. *Suppose that the assumptions of Theorem 13.15 are satisfied. Then for $0 \leqslant s \leqslant 2$, $\tau \in \mathbb{R}$, and $\varepsilon > 0$ we have*

$$\|f^{\varepsilon} e^{-i\tau \mathcal{A}_{\varepsilon}}(f^{\varepsilon})^{-1} - f_0 e^{-i\tau \mathcal{A}^0} f_0^{-1}\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_3(s; \tau) \varepsilon^{s/2},$$

where

$$\mathfrak{C}_3(s; \tau) = (2\|f\|_{L_{\infty}} \|f^{-1}\|_{L_{\infty}})^{1-s/2} (\mathcal{C}_5 + \mathcal{C}_6 |\tau|)^{s/2}. \qquad (13.25)$$

In particular, for $0 < \varepsilon \leqslant 1$ and $|\tau| = \varepsilon^{-\alpha}$, $0 < \alpha < 1$, we have

$$\|f^{\varepsilon} e^{-i\tau \mathcal{A}_{\varepsilon}}(f^{\varepsilon})^{-1} - f_0 e^{-i\tau \mathcal{A}^0} f_0^{-1}\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}_3(s; 1) \varepsilon^{s(1-\alpha)/2},$$
$$0 < \varepsilon \leqslant 1, \quad |\tau| = \varepsilon^{-\alpha}.$$

Theorem 13.16 and Corollary 11.7 imply the following statement.

Corollary 13.17. *Suppose that the matrices $b(\boldsymbol{\theta})$, $g(\mathbf{x})$, and $Q(\mathbf{x}) = (f(\mathbf{x})f(\mathbf{x})^*)^{-1}$ have real entries. Suppose that the spectrum of the generalized spectral problem (10.7) is simple for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. Then the statements of Theorem 13.16 are valid.*

13.8. The sharpness of the result. Applying Theorem 12.8, we confirm the sharpness of the result of Theorem 13.10 in the general case.

Theorem 13.18. *Suppose that the assumptions of Theorem 13.10 are satisfied. Let $\widehat{N}_{0,Q}(\boldsymbol{\theta})$ be defined in Subsection 10.3. Suppose that $\widehat{N}_{0,Q}(\boldsymbol{\theta}_0) \neq 0$ for some $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$. Let $0 \neq \tau \in \mathbb{R}$. Then for any $1 \leqslant s < 3$ it is impossible that the estimate*

$$\| \left(f^\varepsilon e^{-i\tau \mathcal{A}_\varepsilon} (f^\varepsilon)^{-1} - f_0 e^{-i\tau \mathcal{A}^0} f_0^{-1} \right) (\mathcal{H}_0 + I)^{-s/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \mathcal{C}(\tau)\varepsilon$$
(13.26)

holds for all sufficiently small $\varepsilon > 0$.

Proof. We prove by contradiction. Let us fix $\tau \neq 0$. Suppose that for some $1 \leqslant s < 3$ there exists a constant $\mathcal{C}(\tau)$ such that estimate (13.26) holds for all sufficiently small ε. Then, by (13.18), estimate (12.6) also holds. But this contradicts the statement of Theorem 12.8. $\bullet$

§14. Homogenization of the Cauchy problem for the Schrödinger type equation

14.1. The Cauchy problem for the homogeneous equation with the operator $\widehat{\mathcal{A}}_\varepsilon$. Let $\mathbf{u}_\varepsilon(\mathbf{x}, \tau)$, $\mathbf{x} \in \mathbb{R}^d$, $\tau \in \mathbb{R}$, be the solution of the Cauchy problem

$$i\frac{\partial \mathbf{u}_\varepsilon(\mathbf{x}, \tau)}{\partial \tau} = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) \mathbf{u}_\varepsilon(\mathbf{x}, \tau), \quad \mathbf{u}_\varepsilon(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \qquad (14.1)$$

where $\boldsymbol{\phi} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ is a given function. The solution can be represented as $\mathbf{u}_\varepsilon(\cdot, \tau) = e^{-i\tau \widehat{\mathcal{A}}_\varepsilon} \boldsymbol{\phi}$. Let $\mathbf{u}_0(\mathbf{x}, \tau)$ be the solution of the "homogenized" Cauchy problem

$$i\frac{\partial \mathbf{u}_0(\mathbf{x}, \tau)}{\partial \tau} = b(\mathbf{D})^* g^0 b(\mathbf{D}) \mathbf{u}_0(\mathbf{x}, \tau), \quad \mathbf{u}_0(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \qquad (14.2)$$

where g^0 is the effective matrix. Then $\mathbf{u}_0 = e^{-i\tau \widehat{\mathcal{A}}^0} \boldsymbol{\phi}$.

Theorem 13.2 directly implies the following result which has been proved before in [BSu5, Theorem 14.1].

Theorem 14.1. *Let $\mathbf{u}_\varepsilon$ be the solution of problem (14.1), and let $\mathbf{u}_0$ be the solution of problem (14.2).*
1°. If $\boldsymbol{\phi} \in H^s(\mathbb{R}^d; \mathbb{C}^n)$, $0 \leqslant s \leqslant 3$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\|\mathbf{u}_\varepsilon(\cdot, \tau) - \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/3} \widehat{\mathfrak{C}}_1(s; \tau) \|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)}.$$

In particular, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < 1$,

$$\|\mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s(1-\alpha)/3} \widehat{\mathfrak{C}}_1(s; 1) \|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)}. \quad (14.3)$$

The constant $\widehat{\mathfrak{C}}_1(s;\tau)$ is defined by (13.11).

$2°$. If $\phi \in L_2(\mathbb{R}^d;\mathbb{C}^n)$, then

$$\lim_{\varepsilon\to 0}\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R};$$

$$\lim_{\varepsilon\to 0}\|\mathbf{u}_\varepsilon(\cdot,\pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot,\pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} = 0, \quad 0 < \alpha < 1.$$

Statement $2°$ follows directly from statement $1°$ and the Banach-Steinhaus theorem.

Statement $1°$ can be refined under the additional assumptions. Theorem 13.4 implies the following statement.

Theorem 14.2. *Suppose that the assumptions of Theorem* 14.1 *are satisfied. Let $\widehat{N}(\boldsymbol{\theta})$ be the operator defined by* (8.18), (8.19). *Suppose that $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. If $\phi \in H^s(\mathbb{R}^d;\mathbb{C}^n)$, $0 \leqslant s \leqslant 2$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2}\widehat{\mathfrak{C}}_2(s;\tau)\|\phi\|_{H^s(\mathbb{R}^d)}.$$

In particular, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < 1$,

$$\|\mathbf{u}_\varepsilon(\cdot,\pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot,\pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s(1-\alpha)/2}\widehat{\mathfrak{C}}_2(s;1)\|\phi\|_{H^s(\mathbb{R}^d)}. \quad (14.4)$$

The constant $\widehat{\mathfrak{C}}_2(s;\tau)$ is given by (13.14).

Finally, Theorem 13.7 implies the following result.

Theorem 14.3. *Suppose that the assumptions of Theorem* 14.1 *are satisfied. Suppose also that Condition* 9.3 (*or more restrictive Condition* 9.4) *is satisfied. If $\phi \in H^s(\mathbb{R}^d;\mathbb{C}^n)$, $0 \leqslant s \leqslant 2$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|\mathbf{u}_\varepsilon(\cdot,\tau) - \mathbf{u}_0(\cdot,\tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2}\widehat{\mathfrak{C}}_3(s;\tau)\|\phi\|_{H^s(\mathbb{R}^d)}.$$

In particular, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < 1$,

$$\|\mathbf{u}_\varepsilon(\cdot,\pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot,\pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s(1-\alpha)/2}\widehat{\mathfrak{C}}_3(s;1)\|\phi\|_{H^s(\mathbb{R}^d)}.$$

The constant $\widehat{\mathfrak{C}}_3(s;\tau)$ is defined by (13.16).

14.2. The Cauchy problem for the nonhomogeneous equation with the operator $\widehat{\mathcal{A}}_\varepsilon$. Now we consider the Cauchy problem for the nonhomogeneous equation

$$i\frac{\partial\mathbf{u}_\varepsilon(\mathbf{x},\tau)}{\partial\tau} = b(\mathbf{D})^* g^\varepsilon(\mathbf{x})b(\mathbf{D})\mathbf{u}_\varepsilon(\mathbf{x},\tau) + \mathbf{F}(\mathbf{x},\tau), \quad \mathbf{u}_\varepsilon(\mathbf{x},0) = \phi(\mathbf{x}), \quad (14.5)$$

where $\phi \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R};L_2(\mathbb{R}^d;\mathbb{C}^n))$ are given functions. The solution of problem (14.5) can be represented as

$$\mathbf{u}_\varepsilon(\cdot,\tau) = e^{-i\tau\widehat{\mathcal{A}}_\varepsilon}\phi - i\int_0^\tau e^{-i(\tau-\widetilde{\tau})\widehat{\mathcal{A}}_\varepsilon}\mathbf{F}(\cdot,\widetilde{\tau})\,d\widetilde{\tau}. \quad (14.6)$$

Let $\mathbf{u}_0(\mathbf{x},\tau)$ be the solution of the homogenized problem

$$i\frac{\partial\mathbf{u}_0(\mathbf{x},\tau)}{\partial\tau} = b(\mathbf{D})^* g^0 b(\mathbf{D})\mathbf{u}_0(\mathbf{x},\tau) + \mathbf{F}(\mathbf{x},\tau), \quad \mathbf{u}_0(\mathbf{x},0) = \phi(\mathbf{x}). \quad (14.7)$$

Then

$$\mathbf{u}_0(\cdot, \tau) = e^{-i\tau\widehat{\mathcal{A}}^0}\boldsymbol{\phi} - i\int_0^\tau e^{-i(\tau-\widetilde{\tau})\widehat{\mathcal{A}}^0}\mathbf{F}(\cdot, \widetilde{\tau})\, d\widetilde{\tau}. \qquad (14.8)$$

The following theorem has been proved before in [BSu5, Theorem 14.2].

Theorem 14.4. *Let $\mathbf{u}_\varepsilon$ be the solution of problem (14.5), and let $\mathbf{u}_0$ be the solution of problem (14.7).*
$1°$. *If $\boldsymbol{\phi} \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$ with some $0 \leqslant s \leqslant 3$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|\mathbf{u}_\varepsilon(\cdot, \tau) - \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/3}\widehat{\mathfrak{C}}_1(s; \tau)\left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau); H^s(\mathbb{R}^d))}\right).$$
$$(14.9)$$

Under the additional assumption that $\mathbf{F} \in L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d; \mathbb{C}^n))$, where $p \in [1, \infty]$, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < s(s + 3/p')^{-1}$, we have

$$\|\mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)}$$
$$\leqslant \varepsilon^{s(1-\alpha)/3}\widehat{\mathfrak{C}}_1(s; 1)\left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \varepsilon^{-\alpha/p'}\|\mathbf{F}\|_{L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d))}\right). \qquad (14.10)$$

The constant $\widehat{\mathfrak{C}}_1(s; \tau)$ is defined by (13.11). Here $p^{-1} + (p')^{-1} = 1$.
$2°$. *If $\boldsymbol{\phi} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$, then*

$$\lim_{\varepsilon \to 0}\|\mathbf{u}_\varepsilon(\cdot, \tau) - \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R}.$$

Under the additional assumption that $\mathbf{F} \in L_1(\mathbb{R}_\pm; L_2(\mathbb{R}^d; \mathbb{C}^n))$, we have

$$\lim_{\varepsilon \to 0}\|\mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} = 0, \quad 0 < \alpha < 1.$$

Proof. If $\boldsymbol{\phi} \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$ with some $0 \leqslant s \leqslant 3$, then relations (13.10), (14.6), and (14.8) imply (14.9).

If $\mathbf{F} \in L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d; \mathbb{C}^n))$ with some $p \in [1, \infty]$, then for $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < s(s + 3/p')^{-1}$, relations (13.12), (14.6), and (14.8) yield (14.10).

If it is known only that $\boldsymbol{\phi} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$, then from the obvious estimate

$$\|\mathbf{u}_\varepsilon(\cdot, \tau) - \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant 2\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + 2\|\mathbf{F}\|_{L_1((0,\tau); L_2(\mathbb{R}^d))}$$

and (14.9), by the Banach-Steinhaus theorem, it follows that $\mathbf{u}_\varepsilon(\cdot, \tau)$ tends to $\mathbf{u}_0(\cdot, \tau)$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ for a fixed $\tau \in \mathbb{R}$.

If $\mathbf{F} \in L_1(\mathbb{R}_\pm; L_2(\mathbb{R}^d; \mathbb{C}^n))$, then for $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < 1$, from the obvious estimate

$$\|\mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} \leqslant 2\|\boldsymbol{\phi}\|_{L_2(\mathbb{R}^d)} + 2\|\mathbf{F}\|_{L_1(\mathbb{R}_\pm; L_2(\mathbb{R}^d))} \quad (14.11)$$

and (14.10) (with $p = 1$), by the Banach-Steinhaus theorem, it follows that the left-hand side of (14.11) tends to zero. ●

Statement $1°$ of Theorem 14.4 can be refined under the additional assumptions. Theorem 13.4 implies the following result.

Theorem 14.5. *Suppose that the assumptions of Theorem 14.4 are satisfied. Let $\widehat{N}(\boldsymbol{\theta})$ be the operator defined by* (8.18), (8.19). *Suppose that $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. If $\boldsymbol{\phi} \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$ with some $0 \leqslant s \leqslant 2$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|\mathbf{u}_\varepsilon(\cdot, \tau) - \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2} \widehat{\mathfrak{C}}_2(s; \tau) \left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau); H^s(\mathbb{R}^d))} \right).$$

Under the additional assumption that $\mathbf{F} \in L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d; \mathbb{C}^n))$, where $p \in [1, \infty]$, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm \varepsilon^{-\alpha}$, $0 < \alpha < s(s + 2/p')^{-1}$, we have

$$\|\mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)}$$
$$\leqslant \varepsilon^{s(1-\alpha)/2} \widehat{\mathfrak{C}}_2(s; 1) \left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \varepsilon^{-\alpha/p'} \|\mathbf{F}\|_{L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d))} \right).$$

The constant $\widehat{\mathfrak{C}}_2(s; \tau)$ is defined by (13.14).

Finally, Theorem 13.7 implies the following result.

Theorem 14.6. *Suppose that the assumptions of Theorem 14.4 are satisfied. Suppose that Condition 9.3 (or more restrictive Condition 9.4) is satisfied. If $\boldsymbol{\phi} \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$ with some $0 \leqslant s \leqslant 2$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|\mathbf{u}_\varepsilon(\cdot, \tau) - \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2} \widehat{\mathfrak{C}}_3(s; \tau) \left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau); H^s(\mathbb{R}^d))} \right).$$

Under the additional assumption that $\mathbf{F} \in L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d; \mathbb{C}^n))$, where $p \in [1, \infty]$, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm \varepsilon^{-\alpha}$, $0 < \alpha < s(s + 2/p')^{-1}$, we have

$$\|\mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - \mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)}$$
$$\leqslant \varepsilon^{s(1-\alpha)/2} \widehat{\mathfrak{C}}_3(s; 1) \left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \varepsilon^{-\alpha/p'} \|\mathbf{F}\|_{L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d))} \right).$$

The constant $\widehat{\mathfrak{C}}_3(s; \tau)$ is defined by (13.16).

14.3. The Cauchy problem for the homogeneous equation with the operator $\mathcal{A}_\varepsilon$. Let $\mathcal{A}_\varepsilon$ be the operator (13.1). Consider the Cauchy problem

$$i\frac{\partial \mathbf{u}_\varepsilon(\mathbf{x}, \tau)}{\partial \tau} = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}) \mathbf{u}_\varepsilon(\mathbf{x}, \tau), \quad f^\varepsilon(\mathbf{x}) \mathbf{u}_\varepsilon(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}),$$
$$(14.12)$$

where $\boldsymbol{\phi} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$. The solution of problem (14.12) can be represented as $\mathbf{u}_\varepsilon(\cdot, \tau) = e^{-i\tau \mathcal{A}_\varepsilon} (f^\varepsilon)^{-1} \boldsymbol{\phi}$. Let $\mathbf{u}_0(\mathbf{x}, \tau)$ be the solution of the "homogenized" Cauchy problem

$$i\frac{\partial \mathbf{u}_0(\mathbf{x}, \tau)}{\partial \tau} = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D}) f_0 \mathbf{u}_0(\mathbf{x}, \tau), \quad f_0 \mathbf{u}_0(\mathbf{x}, 0) = \boldsymbol{\phi}(\mathbf{x}), \qquad (14.13)$$

where g^0 is the effective matrix (8.9) and f_0 is defined by (10.1). Then $\mathbf{u}_0 = e^{-i\tau \mathcal{A}^0} f_0^{-1} \boldsymbol{\phi}$.

Theorem 13.11 implies the following result (which has been proved before in [BSu5, Theorem 14.3]).

Theorem 14.7. *Let $\mathbf{u}_\varepsilon$ be the solution of problem* (14.12), *and let $\mathbf{u}_0$ be the solution of problem* (14.13).

$1°$. *If $\phi \in H^s(\mathbb{R}^d; \mathbb{C}^n)$, $0 \leqslant s \leqslant 3$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \tau) - f_0 \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/3} \mathfrak{C}_1(s; \tau) \|\phi\|_{H^s(\mathbb{R}^d)}.$$

In particular, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm \varepsilon^{-\alpha}$, $0 < \alpha < 1$,

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \pm \varepsilon^{-\alpha}) - f_0 \mathbf{u}_0(\cdot, \pm \varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s(1-\alpha)/3} \mathfrak{C}_1(s; 1) \|\phi\|_{H^s(\mathbb{R}^d)}.$$

The constant $\mathfrak{C}_1(s; \tau)$ is defined by (13.21).
$2°$. *If $\phi \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, then*

$$\lim_{\varepsilon \to 0} \|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \tau) - f_0 \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R};$$

$$\lim_{\varepsilon \to 0} \|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \pm \varepsilon^{-\alpha}) - f_0 \mathbf{u}_0(\cdot, \pm \varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} = 0, \quad 0 < \alpha < 1.$$

Statement $1°$ of Theorem 14.7 can be refined under the additional assumptions. Theorem 13.13 implies the following result.

Theorem 14.8. *Suppose that the assumptions of Theorem 14.7 are satisfied. Let $\widehat{N}_Q(\boldsymbol{\theta})$ be the operator defined by* (10.10), (10.11). *Suppose that $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. If $\phi \in H^s(\mathbb{R}^d; \mathbb{C}^n)$, $0 \leqslant s \leqslant 2$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \tau) - f_0 \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2} \mathfrak{C}_2(s; \tau) \|\phi\|_{H^s(\mathbb{R}^d)}.$$

In particular, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm \varepsilon^{-\alpha}$, $0 < \alpha < 1$,

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \pm \varepsilon^{-\alpha}) - f_0 \mathbf{u}_0(\cdot, \pm \varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s(1-\alpha)/2} \mathfrak{C}_2(s; 1) \|\phi\|_{H^s(\mathbb{R}^d)}.$$

The constant $\mathfrak{C}_2(s; \tau)$ is defined by (13.23).

Similarly, Theorem 13.16 implies the following result.

Theorem 14.9. *Suppose that the assumptions of Theorem 14.7 are satisfied. Suppose that Condition 11.3 (or more restrictive Condition 11.4) is satisfied. If $\phi \in H^s(\mathbb{R}^d; \mathbb{C}^n)$, $0 \leqslant s \leqslant 2$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \tau) - f_0 \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2} \mathfrak{C}_3(s; \tau) \|\phi\|_{H^s(\mathbb{R}^d)}.$$

In particular, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm \varepsilon^{-\alpha}$, $0 < \alpha < 1$,

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \pm \varepsilon^{-\alpha}) - f_0 \mathbf{u}_0(\cdot, \pm \varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s(1-\alpha)/2} \mathfrak{C}_3(s; 1) \|\phi\|_{H^s(\mathbb{R}^d)}.$$

The constant $\mathfrak{C}_3(s; \tau)$ is defined by (13.25).

14.4. The Cauchy problem for the nonhomogeneous equation with the operator $\mathcal{A}_\varepsilon$. Now we consider the Cauchy problem for the nonhomogeneous equation:

$$i \frac{\partial \mathbf{u}_\varepsilon(\mathbf{x}, \tau)}{\partial \tau} = (f^\varepsilon(\mathbf{x}))^* b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) f^\varepsilon(\mathbf{x}) \mathbf{u}_\varepsilon(\mathbf{x}, \tau) + (f^\varepsilon(\mathbf{x}))^{-1} \mathbf{F}(\mathbf{x}, \tau),$$

$$f^\varepsilon(\mathbf{x}) \mathbf{u}_\varepsilon(\mathbf{x}, 0) = \phi(\mathbf{x}),$$

$$(14.14)$$

where $\phi \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$. The solution of problem (14.14) can be represented as

$$\mathbf{u}_\varepsilon(\cdot, \tau) = e^{-i\tau\mathcal{A}_\varepsilon}(f^\varepsilon)^{-1}\phi - i\int_0^\tau e^{-i(\tau-\widetilde{\tau})\mathcal{A}_\varepsilon}(f^\varepsilon)^{-1}\mathbf{F}(\cdot, \widetilde{\tau})\, d\widetilde{\tau}. \qquad (14.15)$$

Let $\mathbf{u}_0(\mathbf{x}, \tau)$ be the solution of the homogenized problem

$$i\frac{\partial\mathbf{u}_0(\mathbf{x}, \tau)}{\partial\tau} = f_0 b(\mathbf{D})^* g^0 b(\mathbf{D}) f_0 \mathbf{u}_0(\mathbf{x}, \tau) + f_0^{-1}\mathbf{F}(\mathbf{x}, \tau), \quad f_0\mathbf{u}_0(\mathbf{x}, 0) = \phi(\mathbf{x}).$$
$$(14.16)$$

Then

$$\mathbf{u}_0(\cdot, \tau) = e^{-i\tau\mathcal{A}^0}f_0^{-1}\phi - i\int_0^\tau e^{-i(\tau-\widetilde{\tau})\mathcal{A}^0}f_0^{-1}\mathbf{F}(\cdot, \widetilde{\tau})\, d\widetilde{\tau}. \qquad (14.17)$$

By analogy with the proof of Theorem 14.4, from Theorem 13.11 and relations (14.15), (14.17) we deduce the following result (which has been proved before in [BSu5, Theorem 14.5]).

Theorem 14.10. *Let $\mathbf{u}_\varepsilon$ be the solution of problem (14.14), and let $\mathbf{u}_0$ be the solution of problem (14.16).*
1°. If $\phi \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$ with some $0 \leqslant s \leqslant 3$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\|f^\varepsilon\mathbf{u}_\varepsilon(\cdot, \tau) - f_0\mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/3}\mathfrak{C}_1(s; \tau)\left(\|\phi\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau);H^s(\mathbb{R}^d))}\right).$$

Under the additional assumption that $\mathbf{F} \in L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d; \mathbb{C}^n))$, where $p \in [1, \infty]$, for $0 < \varepsilon \leqslant 1$ and $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < s(s + 3/p')^{-1}$, we have

$$\|f^\varepsilon\mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - f_0\mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)}$$
$$\leqslant \varepsilon^{s(1-\alpha)/3}\mathfrak{C}_1(s; 1)\left(\|\phi\|_{H^s(\mathbb{R}^d)} + \varepsilon^{-\alpha/p'}\|\mathbf{F}\|_{L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d))}\right).$$

The constant $\mathfrak{C}_1(s; \tau)$ is defined by (13.21).
2°. If $\phi \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in L_{1,\mathrm{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$, then

$$\lim_{\varepsilon\to 0}\|f^\varepsilon\mathbf{u}_\varepsilon(\cdot, \tau) - f_0\mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R}.$$

Under the additional assumption that $\mathbf{F} \in L_1(\mathbb{R}_\pm; L_2(\mathbb{R}^d; \mathbb{C}^n))$, we have

$$\lim_{\varepsilon\to 0}\|f^\varepsilon\mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - f_0\mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)} = 0, \quad 0 < \alpha < 1.$$

Statement 1° of Theorem 14.10 can be refined under the additional assumptions. Theorem 13.13 implies the following result.

Theorem 14.11. *Suppose that the assumptions of Theorem 14.10 are satisfied. Let $\widehat{N}_Q(\boldsymbol{\theta})$ be the operator defined by (10.10), (10.11). Suppose that $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. If $\phi \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ and $\mathbf{F} \in$*

$L_{1,\text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$ *with some* $0 \leqslant s \leqslant 2$, *then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \tau) - f_0 \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2} \mathfrak{C}_2(s; \tau) \left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau); H^s(\mathbb{R}^d))} \right).$$

Under the additional assumption that $\mathbf{F} \in L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d; \mathbb{C}^n))$, *where* $p \in [1, \infty]$, *for* $0 < \varepsilon \leqslant 1$ *and* $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < s(s + 2/p')^{-1}$, *we have*

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - f_0 \mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)}$$
$$\leqslant \varepsilon^{s(1-\alpha)/2} \mathfrak{C}_2(s; 1) \left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \varepsilon^{-\alpha/p'} \|\mathbf{F}\|_{L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d))} \right).$$

The constant $\mathfrak{C}_2(s; \tau)$ *is defined by* (13.23).

Similarly, applying Theorem 13.16, we deduce the following result.

Theorem 14.12. *Suppose that the assumptions of Theorem* 14.10 *are satisfied. Suppose also that Condition* 11.3 (*or more restrictive Condition* 11.4) *is satisfied. If* $\boldsymbol{\phi} \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ *and* $\mathbf{F} \in L_{1,\text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$ *with some* $0 \leqslant s \leqslant 2$, *then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \tau) - f_0 \mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2} \mathfrak{C}_3(s; \tau) \left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,\tau); H^s(\mathbb{R}^d))} \right).$$

Under the additional assumption that $\mathbf{F} \in L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d; \mathbb{C}^n))$, *where* $p \in [1, \infty]$, *for* $0 < \varepsilon \leqslant 1$ *and* $\tau = \pm\varepsilon^{-\alpha}$, $0 < \alpha < s(s + 2/p')^{-1}$, *we have*

$$\|f^\varepsilon \mathbf{u}_\varepsilon(\cdot, \pm\varepsilon^{-\alpha}) - f_0 \mathbf{u}_0(\cdot, \pm\varepsilon^{-\alpha})\|_{L_2(\mathbb{R}^d)}$$
$$\leqslant \varepsilon^{s(1-\alpha)/2} \mathfrak{C}_3(s; 1) \left(\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^d)} + \varepsilon^{-\alpha/p'} \|\mathbf{F}\|_{L_p(\mathbb{R}_\pm; H^s(\mathbb{R}^d))} \right).$$

The constant $\mathfrak{C}_3(s; \tau)$ *is defined by* (13.25).

§15. APPLICATION OF THE GENERAL RESULTS: THE NONSTATIONARY SCHRÖDINGER EQUATION

15.1. The model example: the Schrödinger type equation with the operator $\widehat{\mathcal{A}}_\varepsilon = -\text{div}\, g^\varepsilon(\mathbf{x})\nabla$. In $L_2(\mathbb{R}^d)$, $d \geqslant 1$, we consider the operator

$$\widehat{\mathcal{A}} = \mathbf{D}^* g(\mathbf{x}) \mathbf{D} = -\text{div}\, g(\mathbf{x})\nabla. \tag{15.1}$$

Here $g(\mathbf{x})$ is a Γ-periodic Hermitian $(d \times d)$-matrix-valued function such that

$$g(\mathbf{x}) > 0; \quad g, g^{-1} \in L_\infty.$$

The operator (15.1) is a particular case of the operator (8.1). We have $n = 1$, $m = d$, and $b(\mathbf{D}) = \mathbf{D}$. Obviously, condition (6.2) is satisfied with $\alpha_0 = \alpha_1 = 1$. According to (8.11), the effective operator for the operator (15.1) is given by

$$\widehat{\mathcal{A}}^0 = \mathbf{D}^* g^0 \mathbf{D} = -\text{div}\, g^0 \nabla.$$

By the general rule, the effective matrix g^0 is defined as follows. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the standard orthonormal basis in $\mathbb{R}^d$. Let $\Phi_j \in \widetilde{H}^1(\Omega)$ be the weak Γ-periodic solution of the problem

$$\operatorname{div} g(\mathbf{x})(\nabla \Phi_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \Phi_j(\mathbf{x}) \, d\mathbf{x} = 0. \tag{15.2}$$

Then g^0 is the $(d \times d)$-matrix with the columns

$$\mathbf{g}_j^0 = |\Omega|^{-1} \int_\Omega g(\mathbf{x})(\nabla \Phi_j(\mathbf{x}) + \mathbf{e}_j) \, d\mathbf{x}, \quad j = 1, \ldots, d.$$

If $d = 1$, then $m = n = 1$, whence $g^0 = \underline{g}$.

If $g(\mathbf{x})$ is a symmetric matrix with real entries, then, by Proposition 8.4(1°), $\widehat{N}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. If $g(\mathbf{x})$ is a Hermitian matrix with complex entries, then, in general, $\widehat{N}(\boldsymbol{\theta})$ is not zero. Now $n = 1$, therefore, $\widehat{N}(\boldsymbol{\theta}) = \widehat{N}_0(\boldsymbol{\theta})$ is the operator of multiplication by $\widehat{\mu}(\boldsymbol{\theta})$, where $\widehat{\mu}(\boldsymbol{\theta})$ is the coefficient in the expansion for the first eigenvalue $\widehat{\lambda}(t, \boldsymbol{\theta}) = \widehat{\gamma}(\boldsymbol{\theta})t^2 + \widehat{\mu}(\boldsymbol{\theta})t^3 + \ldots$ of the operator $\widehat{\mathcal{A}}(\mathbf{k}) = \widehat{A}(t, \boldsymbol{\theta})$. Calculation (see [BSu3, Subsection 10.3]) shows that

$$\widehat{N}(\boldsymbol{\theta}) = \widehat{\mu}(\boldsymbol{\theta}) = -i \sum_{j,l,k=1}^d (a_{jlk} - a_{jlk}^*)\theta_j \theta_l \theta_k, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1},$$

$$a_{jlk} := |\Omega|^{-1} \int_\Omega \Phi_j(\mathbf{x})^* \langle g(\mathbf{x})(\nabla \Phi_l(\mathbf{x}) + \mathbf{e}_l), \mathbf{e}_k \rangle \, d\mathbf{x}, \quad j, l, k = 1, \ldots, d.$$

$$\tag{15.3}$$

The following example is borrowed from [BSu3, Subsection 10.4].

Example 15.1. Let $d = 2$ and $\Gamma = (2\pi\mathbb{Z})^2$. Suppose that the matrix $g(\mathbf{x})$ is given by

$$g(\mathbf{x}) = \begin{pmatrix} 1 & i\beta'(x_1) \\ -i\beta'(x_1) & 1 \end{pmatrix},$$

where $\beta(x_1)$ is a smooth (2π)-periodic real-valued function such that $\int_0^{2\pi} \beta(x_1) \, dx_1 = 0$ and $1 - (\beta'(x_1))^2 > 0$. Then $\widehat{N}(\boldsymbol{\theta}) = -\alpha \pi^{-1}\theta_2^3$, where $\alpha = \int_0^{2\pi} \beta(x_1)(\beta'(x_1))^2 \, dx_1$. It is easy to give a concrete example where $\alpha \neq 0$: if $\beta(x_1) = c(\sin x_1 + \cos 2x_1)$ with $0 < c < 1/3$, then $\alpha = -(3\pi/2)c^3 \neq 0$. In this example, $\widehat{N}(\boldsymbol{\theta}) = \widehat{\mu}(\boldsymbol{\theta}) \neq 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^1$ except for the points $(\pm 1, 0)$.

Consider the Cauchy problem

$$i\frac{\partial u_\varepsilon(\mathbf{x}, \tau)}{\partial \tau} = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}u_\varepsilon(\mathbf{x}, \tau), \quad u_\varepsilon(\mathbf{x}, 0) = \phi(\mathbf{x}), \tag{15.4}$$

where $\phi \in L_2(\mathbb{R}^d)$ is a given function. Let $u_0(\mathbf{x}, \tau)$ be the solution of the "homogenized" Cauchy problem

$$i\frac{\partial u_0(\mathbf{x}, \tau)}{\partial \tau} = \mathbf{D}^* g^0 \mathbf{D} u_0(\mathbf{x}, \tau), \quad u_0(\mathbf{x}, 0) = \phi(\mathbf{x}). \tag{15.5}$$

Applying Theorem 14.1 and, in the "real" case, applying Theorem 14.2, we arrive at the following statement.

Proposition 15.2. *Suppose that the assumptions of Subsection* 15.1 *are satisfied. Let u_ε be the solution of problem* (15.4), *and let u_0 be the solution of problem* (15.5).
1°. *If $\phi \in H^s(\mathbb{R}^d)$ for some $0 \leqslant s \leqslant 3$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/3} \widehat{\mathfrak{C}}_1(s; \tau) \|\phi\|_{H^s(\mathbb{R}^d)},$$

where the constant $\widehat{\mathfrak{C}}_1(s; \tau)$ is given by (13.11). *If $\phi \in L_2(\mathbb{R}^d)$, then*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R}.$$

2°. *Let $g(\mathbf{x})$ be a symmetric matrix with real entries. If $\phi \in H^s(\mathbb{R}^d)$ with some $0 \leqslant s \leqslant 2$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2} \widehat{\mathfrak{C}}_2(s; \tau) \|\phi\|_{H^s(\mathbb{R}^d)},$$

where the constant $\widehat{\mathfrak{C}}_2(s; \tau)$ is given by (13.14).

One can also apply the statements about the behavior of the solution for $|\tau| = \varepsilon^{-\alpha}$ with $0 < \alpha < 1$ (estimates of the form (14.3) in the general case and (14.4) in the "real" case). It is possible to consider more general problem for the nonhomogeneous equation and apply Theorem 14.4 in the general case and Theorem 14.5 in the "real" case.

15.2. The periodic Schrödinger operator. Factorization. (See [BSu1, Chapter 6, Subsection 1.1].) In $L_2(\mathbb{R}^d)$, $d \geqslant 1$, we consider the Schrödinger operator

$$\mathcal{H} = \mathbf{D}^* \breve{g}(\mathbf{x}) \mathbf{D} + V(\mathbf{x}) \tag{15.6}$$

with the Γ-periodic metric $\breve{g}(\mathbf{x})$ and potential $V(\mathbf{x})$. It is assumed that $\breve{g}(\mathbf{x})$ is a symmetric $(d \times d)$-matrix-valued function with real entries, $V(\mathbf{x})$ is a real-valued function, and

$$\breve{g}(\mathbf{x}) > 0; \quad \breve{g}, \breve{g}^{-1} \in L_\infty,$$

$$V \in L_q(\Omega), \quad 2q > d \text{ for } d \geqslant 2; \quad q = 1 \text{ for } d = 1. \tag{15.7}$$

The precise definition of the operator $\mathcal{H}$ is given in terms of the quadratic form

$$h[u, u] = \int_{\mathbb{R}^d} \left(\langle \breve{g}(\mathbf{x}) \mathbf{D} u, \mathbf{D} u \rangle + V(\mathbf{x}) |u|^2 \right) d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d), \tag{15.8}$$

which, under our assumptions, is closed and lower semibounded. Adding an appropriate constant to $V(\mathbf{x})$, we assume that the *point $\lambda_0 = 0$ is the bottom of the spectrum of $\mathcal{H}$.*

Under our assumptions, the equation $\mathbf{D}^*\check{g}(\mathbf{x})\mathbf{D}\omega(\mathbf{x}) + V(\mathbf{x})\omega(\mathbf{x}) = 0$ has a positive Γ-periodic solution $\omega \in \widetilde{H}^1(\Omega)$. Moreover, ω is a multiplier in $H^1(\mathbb{R}^d)$ and $\widetilde{H}^1(\Omega)$. We fix the choice of ω by the normalization condition $\int_\Omega \omega^2(\mathbf{x})\,d\mathbf{x} = |\Omega|$. After the substitution $u = \omega v$, the form (15.8) turns into

$$h[u,u] = \int\limits_{\mathbb{R}^d} \omega^2(\mathbf{x})\langle \check{g}(\mathbf{x})\mathbf{D}v, \mathbf{D}v\rangle\,d\mathbf{x}, \quad u = \omega v, \quad v \in H^1(\mathbb{R}^d).$$

This means that the operator (15.6) admits the following factorization

$$\mathcal{H} = \omega^{-1}\mathbf{D}^*\omega^2\check{g}\mathbf{D}\omega^{-1}. \tag{15.9}$$

Thus, the operator $\mathcal{H}$ is represented in the form (6.4) with $n = 1$, $m = d$, $b(\mathbf{D}) = \mathbf{D}$, $g = \omega^2\check{g}$, and $f = \omega^{-1}$.

Remark 15.3. The expression (15.9) can be taken as the definition of the operator $\mathcal{H}$ for any Γ-periodic function ω such that $\omega(\mathbf{x}) > 0$; $\omega, \omega^{-1} \in L_\infty$. The form (15.6) can be recovered by the formula $V = -\omega^{-1}(\mathbf{D}^*\check{g}\mathbf{D}\omega)$. The corresponding potential V may be a singular distribution.

The operator (15.9) and the operator (15.1) (with $g = \omega^2\check{g}$) satisfy the identity $\mathcal{H} = \omega^{-1}\widehat{\mathcal{A}}\omega^{-1}$. Let g^0 be the effective matrix for the operator (15.1). The function $Q = (ff^*)^{-1}$ takes the form $Q(\mathbf{x}) = \omega^2(\mathbf{x})$. By the normalization condition for ω, we have $\overline{Q} = 1$ and $f_0 = (\overline{Q})^{-1/2} = 1$. Therefore, the operator (10.3) takes the form

$$\mathcal{H}^0 = \mathbf{D}^*g^0\mathbf{D}.$$

By Proposition 10.1(1°), the operator $\widehat{N}_Q(\boldsymbol{\theta})$ is equal to zero: $\widehat{N}_Q(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

15.3. The nonstationary Schrödinger equation with a singular potential. Now we consider the operator

$$\mathcal{H}_\varepsilon = (\omega^\varepsilon)^{-1}\mathbf{D}^*g^\varepsilon\mathbf{D}(\omega^\varepsilon)^{-1}, \quad g^\varepsilon = (\omega^\varepsilon)^2\check{g}^\varepsilon. \tag{15.10}$$

Under condition (15.7), the operator (15.10) can be written in the initial terms:

$$\mathcal{H}_\varepsilon = \mathbf{D}^*\check{g}^\varepsilon\mathbf{D} + \varepsilon^{-2}V^\varepsilon. \tag{15.11}$$

Note that the expression (15.11) contains a large factor ε^{-2} at the rapidly oscillating potential V^ε.

We consider the Cauchy problem of the form (14.12):

$$i\frac{\partial u_\varepsilon(\mathbf{x},\tau)}{\partial\tau} = \mathcal{H}_\varepsilon u_\varepsilon(\mathbf{x},\tau), \quad (\omega^\varepsilon(\mathbf{x}))^{-1}u_\varepsilon(\mathbf{x},0) = \phi(\mathbf{x}), \tag{15.12}$$

where $\phi \in L_2(\mathbb{R}^d)$. Let $u_0(\mathbf{x},\tau)$ be the solution of the homogenized problem (see (14.13))

$$i\frac{\partial u_0(\mathbf{x},\tau)}{\partial\tau} = \mathcal{H}^0 u_0(\mathbf{x},\tau), \quad u_0(\mathbf{x},0) = \phi(\mathbf{x}). \tag{15.13}$$

Applying Theorem 14.8, we arrive at the following resullt.

Proposition 15.4. *Suppose that the assumptions of Subsections* 15.2 *and* 15.3 *are satisfied. Let* u_ε *be the solution of problem* (15.12), *and let* u_0 *be the solution of problem* (15.13). *If* $\phi \in H^s(\mathbb{R}^d)$ *for some* $0 \leqslant s \leqslant 2$, *then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\|(\omega^\varepsilon)^{-1} u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/2} \mathfrak{C}_2(s; \tau) \|\phi\|_{H^s(\mathbb{R}^d)},$$

where the constant $\mathfrak{C}_2(s; \tau)$ *is given by* (13.23). *If* $\phi \in L_2(\mathbb{R}^d)$, *then*

$$\lim_{\varepsilon \to 0} \|(\omega^\varepsilon)^{-1} u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R}.$$

One can also apply the statement of Theorem 14.8 about the behavior of the solutions for $|\tau| = \varepsilon^{-\alpha}$ with $0 < \alpha < 1$. It is also possible to consider more general Cauchy problem for the nonhomogeneous equation and to apply Theorem 14.11.

15.4. The nonstationary Schrödinger equation with a magnetic potential. In $L_2(\mathbb{R}^d)$, $d \geqslant 2$, we consider the periodic magnetic Schrödinger operator $\mathcal{M}$ with Γ-periodic metric $\check{g}(\mathbf{x})$, magnetic potential $\mathbf{A}(\mathbf{x})$, and electric potential $V(\mathbf{x})$:

$$\mathcal{M} = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^* \check{g}(\mathbf{x})(\mathbf{D} - \mathbf{A}(\mathbf{x})) + V(\mathbf{x}).$$

Here $\check{g}(\mathbf{x})$ is a symmetric $(d \times d)$-matrix-valued function with real entries such that $\check{g}(\mathbf{x}) > 0$ and $\check{g}, \check{g}^{-1} \in L_\infty$. If $d \geqslant 3$, we assume in addition that $\check{g} \in C^\alpha$ with some $0 < \alpha < 1$. Suppose that $\mathbf{A}(\mathbf{x})$ is a $\mathbb{R}^d$-valued function and $V(\mathbf{x})$ is a real-valued function such that

$$\mathbf{A} \in L_{2q}(\Omega), \quad V \in L_q(\Omega), \quad 2q > d.$$

As usual, the precise definition of the operator is given in terms of the corresponding quadratic form. Adding an appropriate constant to $V(\mathbf{x})$, we assume that the bottom of the spectrum of $\mathcal{M}$ is the point $\lambda_0 = 0$.

According to [Sh2], under the above assumptions and for sufficiently small (in the $L_{2q}(\Omega)$-norm) magnetic potential $\mathbf{A}$, the operator $\mathcal{M}$ admits an appropriate factorization. Let us describe this factorization. Let $\mathcal{M}(\mathbf{k})$ be the family of operators in $L_2(\Omega)$ that arise in the direct integral expansion for $\mathcal{M}$. The condition $\inf \operatorname{spec} \mathcal{M} = 0$ means that for some $\mathbf{k}_0 \in \widetilde{\Omega}$ the point $\lambda_0 = 0$ is an eigenvalue of $\mathcal{M}(\mathbf{k}_0)$. If the magnetic potential is sufficiently small, then this point $\mathbf{k}_0$ is unique and the eigenvalue $\lambda_0 = 0$ of the operator $\mathcal{M}(\mathbf{k}_0)$ is simple. Let $\eta(\mathbf{x})$ be the corresponding eigenfunction normalized by the condition $\int_\Omega |\eta(\mathbf{x})|^2 \, d\mathbf{x} = |\Omega|$ (the phase factor is not important). Then $\eta \in \widetilde{H}^1(\Omega)$, and $\eta, \eta^{-1} \in L_\infty$. As shown in [Sh2], η is a multiplier in $H^1(\mathbb{R}^d)$ and $\widetilde{H}^1(\Omega)$. We denote

$$\widetilde{\mathcal{M}} := [e^{-i\langle \mathbf{k}_0, \cdot \rangle}] \mathcal{M} [e^{i\langle \mathbf{k}_0, \cdot \rangle}].$$

Clearly, the coefficients of the operator $\widetilde{\mathcal{M}}$ are periodic. By Theorems 2.7 and 2.8 from [Sh2], if the norm $\|\mathbf{A}\|_{L_{2q}(\Omega)}$ is sufficiently small, then the

operator $\widetilde{\mathcal{M}}$ admits the following factorization:

$$\widetilde{\mathcal{M}} = (\eta(\mathbf{x})^*)^{-1}\mathbf{D}^* g(\mathbf{x})\mathbf{D}\eta(\mathbf{x})^{-1}. \tag{15.14}$$

Here the Hermitian Γ-periodic matrix-valued function $g(\mathbf{x})$ is defined by

$$g(\mathbf{x}) = |\eta(\mathbf{x})|^2 \breve{g}(\mathbf{x}) + ig_2(\mathbf{x}), \tag{15.15}$$

and the antisymmetric matrix-valued function $g_2(\mathbf{x})$ with real entries satisfies the equation

$$(\operatorname{div} g_2(\mathbf{x}))^t = -2|\eta(\mathbf{x})|^2 \breve{g}(\mathbf{x})(\mathbf{A}(\mathbf{x}) - \mathbf{k}_0) + 2\operatorname{Im}(\eta(\mathbf{x})^* \breve{g}(\mathbf{x})\nabla\eta(\mathbf{x})). \tag{15.16}$$

As shown in [Sh2], we have

$$g(\mathbf{x}) > 0; \quad g, g^{-1} \in L_\infty.$$

The operator (15.14) is of the form (6.4) with $n = 1$, $m = d$, $b(\mathbf{D}) = \mathbf{D}$, g defined by (15.15), (15.16), and $f = \eta^{-1}$. Let g^0 be the effective matrix for the operator $\widehat{\mathcal{A}} = \mathbf{D}^* g\mathbf{D}$; in general, the effective matrix may have complex entries. Now the function $Q = (ff^*)^{-1}$ takes the form $Q(\mathbf{x}) = |\eta(\mathbf{x})|^2$. By the normalization condition on η, we have $\overline{Q} = 1$, and then $f_0 = 1$. The operator (10.3) takes the form

$$\widetilde{\mathcal{M}}^0 = \mathbf{D}^* g^0 \mathbf{D}.$$

Let us describe the operator $\widehat{N}_Q(\boldsymbol{\theta})$. Let Φ_j be the Γ-periodic solution of problem (15.2). Since $n = 1$ and $\overline{Q} = 1$, then (see (5.9)) the operator $\widehat{N}_Q(\boldsymbol{\theta}) = \widehat{N}_{0,Q}(\boldsymbol{\theta})$ acts as multiplication by $\mu(\boldsymbol{\theta})$, where $\mu(\boldsymbol{\theta})$ is the coefficient in the expansion $\lambda(t, \boldsymbol{\theta}) = \gamma(\boldsymbol{\theta})t^2 + \mu(\boldsymbol{\theta})t^3 + \dots$ for the first eigenvalue $\lambda(t, \boldsymbol{\theta})$ of the operator $\widetilde{\mathcal{M}}(\mathbf{k}) = \widetilde{\mathcal{M}}(t\boldsymbol{\theta})$. A calculation shows that

$$\widehat{N}_Q(\boldsymbol{\theta}) = \mu(\boldsymbol{\theta}) = -i\sum_{j,l,k=1}^{d}(a_{jlk} - a_{jlk}^*)\theta_j\theta_l\theta_k + 2\langle g^0\boldsymbol{\theta}, \boldsymbol{\theta}\rangle\sum_{j=1}^{d}\operatorname{Im}(\overline{Q\Phi_j})\theta_j,$$
$$\boldsymbol{\theta} \in \mathbb{S}^{d-1},$$

where the coefficients a_{jlk} are defined by (15.3). In general, the operator $\widehat{N}_Q(\boldsymbol{\theta})$ is not zero.

Now we consider the operators

$$\widetilde{\mathcal{M}}_\varepsilon = ((\eta^\varepsilon)^*)^{-1}\mathbf{D}^* g^\varepsilon \mathbf{D}(\eta^\varepsilon)^{-1}, \quad \mathcal{M}_\varepsilon = [e^{i\varepsilon^{-1}\langle\mathbf{k}_0,\cdot\rangle}]\widetilde{\mathcal{M}}_\varepsilon[e^{-i\varepsilon^{-1}\langle\mathbf{k}_0,\cdot\rangle}].$$

In the initial terms, we have

$$\mathcal{M}_\varepsilon = (\mathbf{D} - \varepsilon^{-1}\mathbf{A}^\varepsilon)^* \breve{g}^\varepsilon(\mathbf{D} - \varepsilon^{-1}\mathbf{A}^\varepsilon) + \varepsilon^{-2}V^\varepsilon. \tag{15.17}$$

Note that the expression (15.17) contains large factors ε^{-1} at the rapidly oscillating magnetic potential $\mathbf{A}^\varepsilon$ and ε^{-2} at the electric potential V^ε. Let $u_\varepsilon(\mathbf{x}, \tau)$ be the solution of the Cauchy problem for the nonstationary magnetic Schrödinger equation:

$$i\frac{\partial u_\varepsilon(\mathbf{x}, \tau)}{\partial\tau} = \mathcal{M}_\varepsilon u_\varepsilon(\mathbf{x}, \tau), \quad (\eta^\varepsilon(\mathbf{x}))^{-1}e^{-i\varepsilon^{-1}\langle\mathbf{k}_0,\mathbf{x}\rangle}u_\varepsilon(\mathbf{x}, 0) = \phi(\mathbf{x}), \tag{15.18}$$

where $\phi \in L_2(\mathbb{R}^d)$. Then the function $v_\varepsilon(\mathbf{x}, \tau) = e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \mathbf{x}\rangle} u_\varepsilon(\mathbf{x}, \tau)$ is the solution of the problem

$$i\frac{\partial v_\varepsilon(\mathbf{x}, \tau)}{\partial \tau} = \widetilde{\mathcal{M}}_\varepsilon v_\varepsilon(\mathbf{x}, \tau), \quad (\eta^\varepsilon(\mathbf{x}))^{-1} v_\varepsilon(\mathbf{x}, 0) = \phi(\mathbf{x}). \tag{15.19}$$

We can apply Theorem 14.7. The effective problem is of the form

$$i\frac{\partial v_0(\mathbf{x}, \tau)}{\partial \tau} = \mathbf{D}^* g^0 \mathbf{D} v_0(\mathbf{x}, \tau), \quad v_0(\mathbf{x}, 0) = \phi(\mathbf{x}). \tag{15.20}$$

Applying Theorem 14.7 to the problem (15.19), we arrive at the following result.

Proposition 15.5. *Suppose that the assumptions of Subsection* 15.4 *are satisfied. Let* u_ε *be the solution of problem* (15.18), *and let* v_0 *be the solution of problem* (15.20). *If* $\phi \in H^s(\mathbb{R}^d)$ *with some* $0 \leqslant s \leqslant 3$, *then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\|(\eta^\varepsilon)^{-1} e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \mathbf{x}\rangle} u_\varepsilon(\cdot, \tau) - v_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon^{s/3} \mathfrak{C}_1(s; \tau) \|\phi\|_{H^s(\mathbb{R}^d)},$$

where the constant $\mathfrak{C}_1(s; \tau)$ *is given by* (13.21). *If* $\phi \in L_2(\mathbb{R}^d)$, *then*

$$\lim_{\varepsilon \to 0} \|(\eta^\varepsilon)^{-1} e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \mathbf{x}\rangle} u_\varepsilon(\cdot, \tau) - v_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R}.$$

One can also apply the statemment of Theorem 14.7 about the behavior of the solutions for $|\tau| = \varepsilon^{-\alpha}$ with $0 < \alpha < 1$. It is also possible to consider more general Cauchy problem for the nonhomogeneous equation and to apply Theorem 14.10.

Remark 15.6. In [Sh1] it was shown that in general (without the smallness condition on $\mathbf{A}$) the required factorization for the magnetic Schrödinger operator is not valid. This leads to interesting effects in the corresponding homogenization problem (see [Sh3]).

§16. APPLICATION OF THE GENERAL RESULTS: THE NONSTATIONARY TWO-DIMENSIONAL PAULI EQUATION

16.1. Definition and factorization of the two-dimensional Pauli operator. (See [BSu1, Chapter 6, Subsection 2.1].) Suppose that the magnetic potential is a vector-valued function $\mathbf{A}(\mathbf{x}) = \{A_1(\mathbf{x}), A_2(\mathbf{x})\}$ in $\mathbb{R}^2$, where $A_j(\mathbf{x})$ are Γ-periodic real-valued functions such that

$$A_j \in L_\rho(\Omega), \quad \rho > 2, \quad j = 1, 2. \tag{16.1}$$

Recall the standard notation for the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In $L_2(\mathbb{R}^2; \mathbb{C}^2)$, we consider the operator

$$\mathcal{D} = (D_1 - A_1)\sigma_1 + (D_2 - A_2)\sigma_2, \quad \text{Dom}\,\mathcal{D} = H^1(\mathbb{R}^2; \mathbb{C}^2). \tag{16.2}$$

By definition, the Pauli operator is the square of $\mathcal{D}$:

$$\mathcal{P} := \mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}. \tag{16.3}$$

Precisely, $\mathcal{P}$ is the selfadjoint operator in $L_2(\mathbb{R}^2; \mathbb{C}^2)$ corresponding to the closed quadratic form $\|\mathcal{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^2)}$, $\mathbf{u} \in H^1(\mathbb{R}^2; \mathbb{C}^2)$. If $\mathbf{A}(\mathbf{x})$ is Lipschitz, then the blocks $P_\pm$ of the operator (16.3) can be written as

$$P_\pm = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^2 \pm B(\mathbf{x}), \quad B(\mathbf{x}) := \partial_1 A_2(\mathbf{x}) - \partial_2 A_1(\mathbf{x}).$$

We use a well known factorization for the two-dimensional Pauli operator. A gauge transformation allows us to assume that the potential $\mathbf{A}$ is subject to the conditions

$$\operatorname{div} \mathbf{A}(\mathbf{x}) = 0, \quad \int_\Omega \mathbf{A}(\mathbf{x})\, d\mathbf{x} = 0, \tag{16.4}$$

and still satisfies (16.1). Under conditions (16.1) and (16.4), there exists a (unique) real-valued Γ-periodic function φ such that

$$\nabla\varphi(\mathbf{x}) = \{A_2(\mathbf{x}), -A_1(\mathbf{x})\}, \quad \int_\Omega \varphi(\mathbf{x})\, d\mathbf{x} = 0.$$

Note that $\varphi \in \widetilde{W}^1_\rho(\Omega) \subset C^\sigma$, $\sigma = 1 - 2\rho^{-1}$. We put

$$\omega_\pm(\mathbf{x}) := e^{\pm\varphi(\mathbf{x})}.$$

The operators (16.2), (16.3) admit the following factorization:

$$\mathcal{D} = f_\times(\mathbf{x}) b_\times(\mathbf{D}) f_\times(\mathbf{x}), \tag{16.5}$$

$$\mathcal{P} = f_\times(\mathbf{x}) b_\times(\mathbf{D}) g_\times(\mathbf{x}) b_\times(\mathbf{D}) f_\times(\mathbf{x}), \tag{16.6}$$

where

$$b_\times(\mathbf{D}) = \begin{pmatrix} 0 & D_1 - iD_2 \\ D_1 + iD_2 & 0 \end{pmatrix},$$

$$f_\times(\mathbf{x}) = \begin{pmatrix} \omega_+(\mathbf{x}) & 0 \\ 0 & \omega_-(\mathbf{x}) \end{pmatrix}, \quad g_\times(\mathbf{x}) = f_\times(\mathbf{x})^2 = \begin{pmatrix} \omega_+^2(\mathbf{x}) & 0 \\ 0 & \omega_-^2(\mathbf{x}) \end{pmatrix}.$$

The blocks $P_\pm$ of the operator (16.3) can be written as

$$P_+ = \omega_-(D_1 + iD_2)\omega_+^2(D_1 - iD_2)\omega_-, \quad P_- = \omega_+(D_1 - iD_2)\omega_-^2(D_1 + iD_2)\omega_+. \tag{16.7}$$

Remark 16.1. 1) We may take expressions (16.5), (16.6), (16.7) as the definition of the operators $\mathcal{D}$, $\mathcal{P}$, and $P_\pm$, assuming that $\omega_\pm(\mathbf{x})$ are arbitrary Γ-periodic functions satisfying the conditions $\omega_\pm(\mathbf{x}) > 0$; $\omega_+, \omega_- \in L_\infty$, and $\omega_+(\mathbf{x})\omega_-(\mathbf{x}) = 1$. 2) Note that the operators P_+ and P_- are unitarily equivalent. Moreover, the operators $P_+(\mathbf{k})$ and $P_-(\mathbf{k})$ in $L_2(\Omega)$ are also unitarily equivalent for each $\mathbf{k}$.

16.2. The effective characteristics for the operators $P_\pm$. Homogenization. The operators $P_\pm$ are of the form (6.4) with $d = 2$, $m = n = 1$,

$b(\mathbf{D}) = D_1 \mp iD_2$, $g(\mathbf{x}) = \omega_\pm^2(\mathbf{x})$, and $f(\mathbf{x}) = \omega_\mp(\mathbf{x})$. The role of the operator $\widehat{\mathcal{A}}$ for $P_\pm$ is played by the operatoor $\widehat{\mathcal{A}}_\pm = (D_1 \pm iD_2)\omega_\pm^2(D_1 \mp iD_2)$. Since $m = n$, the effective constant is given by

$$g_\pm^0 = \underline{\omega_\pm^2} = \left(|\Omega|^{-1} \int_\Omega \omega_\mp^2(\mathbf{x})\, d\mathbf{x} \right)^{-1} =: \omega_{\pm,0}^2. \tag{16.8}$$

The role of $Q(\mathbf{x})$ for the operator $P_\pm$ is played by $Q_\pm(\mathbf{x}) = \omega_\pm^2(\mathbf{x})$. Then, by (16.8), $\overline{Q_\pm} = (g_\mp^0)^{-1}$. The role of f_0 is played by the constant $(\overline{Q_\pm})^{-1/2} = (g_\mp^0)^{1/2} = \omega_{\mp,0}$. Next, the role of $\mathcal{A}^0$ for $P_\pm$ is played by the operator $P_\pm^0$, where

$$P_+^0 = \omega_{-,0}(D_1 + iD_2)g_+^0(D_1 - iD_2)\omega_{-,0} = -\gamma\Delta,$$
$$P_-^0 = \omega_{+,0}(D_1 - iD_2)g_-^0(D_1 + iD_2)\omega_{+,0} = -\gamma\Delta.$$

Here

$$\gamma := g_+^0 g_-^0 = |\Omega|^2 \|\omega_+\|_{L_2(\Omega)}^{-2} \|\omega_-\|_{L_2(\Omega)}^{-2}. \tag{16.9}$$

Let $\lambda_\pm(t, \boldsymbol{\theta})$ be the analytic (in t) branch of the first eigenvalue of the operator $P_\pm(\mathbf{k})$, and let $\lambda_\pm(t, \boldsymbol{\theta}) = \gamma_\pm(\boldsymbol{\theta})t^2 + \mu_\pm(\boldsymbol{\theta})t^3 + \ldots$ be the corresponding power series expansion. Since the operators $P_+(\mathbf{k})$ and $P_-(\mathbf{k})$ are unitarily equivalent, then $\lambda_+(t, \boldsymbol{\theta}) = \lambda_-(t, \boldsymbol{\theta})$, and also $\gamma_+(\boldsymbol{\theta}) = \gamma_-(\boldsymbol{\theta})$, $\mu_+(\boldsymbol{\theta}) = \mu_-(\boldsymbol{\theta})$. As shown in [BSu1, Chapter 6, §2], the numbers $\gamma_\pm(\boldsymbol{\theta})$ do not depend on $\boldsymbol{\theta}$ and are given by

$$\gamma_+(\boldsymbol{\theta}) = \gamma_-(\boldsymbol{\theta}) = \gamma,$$

where γ is defined by (16.9). Now we describe the operator $\widehat{N}_{Q,\pm}(\boldsymbol{\theta})$ that plays the role of $\widehat{N}_Q(\boldsymbol{\theta})$ for $P_\pm$. Let $v_\pm(\mathbf{x})$ be the Γ-periodic solution of the problem

$$(D_1 \mp iD_2)v_\pm(\mathbf{x}) = g_\pm^0 \omega_\mp^2(\mathbf{x}) - 1, \quad \int_\Omega v_\pm(\mathbf{x}) = 0. \tag{16.10}$$

Then

$$\widehat{N}_{Q,\pm}(\boldsymbol{\theta}) = -2\gamma \left(\theta_1 \operatorname{Re} \overline{\omega_\pm^2 v_\pm} \pm \theta_2 \operatorname{Im} \overline{\omega_\pm^2 v_\pm} \right), \quad \boldsymbol{\theta} \in \mathbb{S}^1. \tag{16.11}$$

According to (10.12),

$$\mu_\pm(\boldsymbol{\theta}) = -2g_\mp^0 \gamma \left(\theta_1 \operatorname{Re} \overline{\omega_\pm^2 v_\pm} \pm \theta_2 \operatorname{Im} \overline{\omega_\pm^2 v_\pm} \right), \quad \boldsymbol{\theta} \in \mathbb{S}^1. \tag{16.12}$$

Though we know that $\mu_+(\boldsymbol{\theta}) = \mu_-(\boldsymbol{\theta}) =: \mu(\boldsymbol{\theta})$, it is not evident to deduce this directly from (16.12). Now we give an example where $\mu(\boldsymbol{\theta}) \neq 0$.

Example 16.2. Let $\Gamma = (2\pi\mathbb{Z})^2$ and let $\omega_-^2(\mathbf{x}) = 1 + \alpha(\sin x_2 + 4\sin 2x_2)$, where $\alpha > 0$ is sufficiently small. Then, according to (16.8) and (16.10), we

have $g_+^0 = 1$ and $v_+(\mathbf{x}) = \alpha(\cos x_2 + 2\cos 2x_2)$. Let us calculate $\overline{\omega_+^2 v_+}$:

$$\overline{\omega_+^2 v_+} = \frac{\alpha}{2\pi} \int\limits_0^{2\pi} \frac{\cos x + 2\cos 2x}{1 + \alpha(\sin x + 4\sin 2x)}\, dx = \frac{3\alpha}{8\pi} \int\limits_0^{2\pi} \frac{\cos x}{1 + \alpha(\sin x + 4\sin 2x)}\, dx$$

$$= -\frac{6\alpha^2}{\pi} \int\limits_{-1}^1 \frac{t\sqrt{1 - t^2}}{(1 + \alpha t)^2 - 64\alpha^2 t^2(1 - t^2)}\, dt$$

$$= \frac{24\alpha^3}{\pi} \int\limits_0^1 \frac{t^2\sqrt{1 - t^2}}{\left((1 + \alpha t)^2 - 64\alpha^2 t^2(1 - t^2)\right)\left((1 - \alpha t)^2 - 64\alpha^2 t^2(1 - t^2)\right)}\, dt.$$

Obviously, for sufficiently small $\alpha > 0$ (for instance, one can take $\alpha = \frac{1}{16}$) the function in the last integral is positive, whence $\overline{\omega_+^2 v_+} > 0$. Then $\mu(\boldsymbol{\theta}) = -2g_-^0 \gamma \theta_1 \overline{\omega_+^2 v_+} \neq 0$ for $\theta_1 \neq 0$.

Now we consider the operators

$$\begin{aligned}
P_{+,\varepsilon} &= \omega_-^\varepsilon(D_1 + iD_2)(\omega_+^\varepsilon)^2(D_1 - iD_2)\omega_-^\varepsilon, \\
P_{-,\varepsilon} &= \omega_+^\varepsilon(D_1 - iD_2)(\omega_-^\varepsilon)^2(D_1 + iD_2)\omega_+^\varepsilon.
\end{aligned} \tag{16.13}$$

If $\mathbf{A}$ is Lipschitz, then the operators (16.13) can be written as

$$P_{\pm,\varepsilon} = (\mathbf{D} - \varepsilon^{-1}\mathbf{A}^\varepsilon)^2 \pm \varepsilon^{-2}B^\varepsilon.$$

We consider the following Cauchy problems for the scalar functions $u_{\pm,\varepsilon}(\mathbf{x}, \tau)$:

$$i\frac{\partial u_{\pm,\varepsilon}(\mathbf{x}, \tau)}{\partial \tau} = P_{\pm,\varepsilon}u_{\pm,\varepsilon}(\mathbf{x}, \tau), \quad \omega_\mp^\varepsilon(\mathbf{x})u_{\pm,\varepsilon}(\mathbf{x}, 0) = \phi_\pm(\mathbf{x}), \tag{16.14}$$

where $\phi_\pm \in L_2(\mathbb{R}^2)$. We can apply Theorem 14.7. The corresponding homogenized problems are

$$i\frac{\partial u_{\pm,0}(\mathbf{x}, \tau)}{\partial \tau} = -\gamma\Delta u_{\pm,0}(\mathbf{x}, \tau), \quad \omega_{\mp,0}u_{\pm,0}(\mathbf{x}, 0) = \phi_\pm(\mathbf{x}), \tag{16.15}$$

where γ is defined by (16.9), and $\omega_{\pm,0}$ are given by (16.8).

Proposition 16.3. *Suppose that the assumptions of Subsections* 16.1, 16.2 *are satisfied. Let $u_{\pm,\varepsilon}$ be the solution of problem* (16.14), *and let $u_{\pm,0}$ be the solution of problem* (16.15). *If $\phi_\pm \in H^s(\mathbb{R}^2)$ with some $0 \leqslant s \leqslant 3$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\|\omega_\mp^\varepsilon u_{\pm,\varepsilon}(\cdot, \tau) - \omega_{\mp,0}u_{\pm,0}(\cdot, \tau)\|_{L_2(\mathbb{R}^2)} \leqslant \varepsilon^{s/3}\mathfrak{C}_\pm(s; \tau)\|\phi_\pm\|_{H^s(\mathbb{R}^2)},$$

where the constants $\mathfrak{C}_\pm(s; \tau)$ are of the form (13.21). *If $\phi_\pm \in L_2(\mathbb{R}^2)$, then*

$$\lim_{\varepsilon\to 0} \|\omega_\mp^\varepsilon u_{\pm,\varepsilon}(\cdot, \tau) - \omega_{\mp,0}u_{\pm,0}(\cdot, \tau)\|_{L_2(\mathbb{R}^2)} = 0, \quad \tau \in \mathbb{R}.$$

One can also apply the statement of Theorem 14.7 about the behavior of the solutions for $|\tau| = \varepsilon^{-\alpha}$ with $0 < \alpha < 1$. It is also possible to consider

more general Cauchy problem for the nonhomogeneous equation and apply Theorem 14.10.

16.3. The effective characteristics for the operator $\mathcal{P}$. Homogenization. The operator $\mathcal{P}$ is of the form (6.4) with $d = 2$, $m = n = 2$, $b(\mathbf{D}) = b_\times(\mathbf{D})$, $g(\mathbf{x}) = g_\times(\mathbf{x})$, and $f(\mathbf{x}) = f_\times(\mathbf{x})$. The role of the operator $\widehat{\mathcal{A}}$ for $\mathcal{P}$ is played by the operator $\widehat{\mathcal{A}}_\times = b_\times(\mathbf{D})g_\times(\mathbf{x})b_\times(\mathbf{D})$. Since $m = n$, the effective matrix is given by

$$g_\times^0 = \underline{g_\times} = \operatorname{diag}\{g_+^0, g_-^0\},$$

where $g_\pm^0$ are defined by (16.8). The role of Q is played by the matrix $Q_\times = f_\times^{-2} = g_\times^{-1}$. Then $\overline{Q_\times} = \operatorname{diag}\{(g_+^0)^{-1}, (g_-^0)^{-1}\}$. The role of f_0 is played by the matrix

$$f_{\times,0} = \operatorname{diag}\{\omega_{+,0}, \omega_{-,0}\}. \tag{16.16}$$

Next, the operator $\mathcal{A}^0$ (see (10.3)) takes the form

$$\mathcal{P}^0 = f_{\times,0}b_\times(\mathbf{D})g_\times^0 b_\times(\mathbf{D})f_{\times,0} = \begin{pmatrix} -\gamma\Delta & 0 \\ 0 & -\gamma\Delta \end{pmatrix}.$$

Let us describe the operator $\widehat{N}_{Q,\times}(\boldsymbol{\theta})$ that plays the role of $\widehat{N}_Q(\boldsymbol{\theta})$ for $\mathcal{P}$. A calculation shows that

$$\widehat{N}_{Q,\times}(\boldsymbol{\theta}) = \operatorname{diag}\{\widehat{N}_{Q,+}(\boldsymbol{\theta}), \widehat{N}_{Q,-}(\boldsymbol{\theta})\},$$

where the operators $\widehat{N}_{Q,\pm}(\boldsymbol{\theta})$ are defined by (16.11). The first eigenvalue $\lambda(t, \boldsymbol{\theta})$ (analytic in t) of the operator $\mathcal{P}(\mathbf{k})$ is of multiplicity two identically in $\mathbf{k} = t\boldsymbol{\theta}$, because the blocks $P_+(\mathbf{k})$ and $P_-(\mathbf{k})$ are unitarily equivalent. In the power series expansion $\lambda(t, \boldsymbol{\theta}) = \gamma t^2 + \mu(\boldsymbol{\theta})t^3 + \ldots$, the coefficient γ is independent of $\boldsymbol{\theta}$ and defined by (16.9), and the coefficient $\mu(\boldsymbol{\theta})$ is defined by (16.12). Example 16.2 shows that, in the general case, the coefficient $\mu(\boldsymbol{\theta})$ is not zero.

Now we consider the operator

$$\mathcal{P}_\varepsilon = f_\times^\varepsilon b_\times(\mathbf{D})g_\times^\varepsilon b_\times(\mathbf{D})f_\times^\varepsilon = \begin{pmatrix} P_{-,\varepsilon} & 0 \\ 0 & P_{+,\varepsilon} \end{pmatrix},$$

where the blocks are defined by (16.13). Consider the Cauchy problem for a vector-valued function $\mathbf{u}_\varepsilon(\mathbf{x}, \tau)$:

$$i\frac{\partial \mathbf{u}_\varepsilon(\mathbf{x}, \tau)}{\partial \tau} = \mathcal{P}_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x}, \tau), \quad f_\times^\varepsilon(\mathbf{x})\mathbf{u}_\varepsilon(\mathbf{x}, 0) = \phi(\mathbf{x}), \tag{16.17}$$

where $\phi \in L_2(\mathbb{R}^2; \mathbb{C}^2)$. Let $\phi = \operatorname{col}\{\phi_-, \phi_+\}$. Clearly, we have $\mathbf{u} = \operatorname{col}\{u_{-,\varepsilon}, u_{+,\varepsilon}\}$, where $u_{\pm,\varepsilon}$ are the solutions of problems (16.14).

We can apply Theorem 14.7 to problem (16.17). The corresponding homogenized problem is

$$i\frac{\partial \mathbf{u}_0(\mathbf{x}, \tau)}{\partial \tau} = -\gamma\Delta\mathbf{u}_0(\mathbf{x}, \tau), \quad f_{\times,0}\mathbf{u}_0(\mathbf{x}, 0) = \phi(\mathbf{x}), \tag{16.18}$$

where γ is defined by (16.9), and the matrix $f_{\times,0}$ is defined by (16.16). Clearly, we have $\mathbf{u}_0 = \mathrm{col}\,\{u_{-,0}, u_{+,0}\}$, where $u_{\pm,0}$ are the solutions of problems (16.15).

Proposition 16.4. *Supose that the assumptions of Subsections* 16.1–16.3 *are satisfied. Let* $\mathbf{u}_\varepsilon$ *be the solution of problem* (16.17), *and let* $\mathbf{u}_0$ *be the solution of problem* (16.18). *If* $\boldsymbol{\phi} \in H^s(\mathbb{R}^2; \mathbb{C}^2)$ *with some* $0 \leqslant s \leqslant 3$, *then for* $\tau \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$\|f_\times^\varepsilon \mathbf{u}_\varepsilon(\cdot, \tau) - f_{\times,0}\mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^2)} \leqslant \varepsilon^{s/3}\mathfrak{C}_\times(s; \tau)\|\boldsymbol{\phi}\|_{H^s(\mathbb{R}^2)}.$$

The constant $\mathfrak{C}_\times(s; \tau)$ *is given by* (13.21). *If* $\boldsymbol{\phi} \in L_2(\mathbb{R}^2; \mathbb{C}^2)$, *then*

$$\lim_{\varepsilon \to 0} \|f_\times^\varepsilon \mathbf{u}_\varepsilon(\cdot, \tau) - f_{\times,0}\mathbf{u}_0(\cdot, \tau)\|_{L_2(\mathbb{R}^2)} = 0, \quad \tau \in \mathbb{R}.$$

One can also apply the statement of Theorem 14.7 about the behavior of the solution for $|\tau| = \varepsilon^{-\alpha}$ with $0 < \alpha < 1$. Also it is possible to consider more general Cauchy problem for the nonhomogeneous equation and apply Theorem 14.10.

References

[BaPa]   Bakhvalov N. S., Panasenko G. P., *Homogenization: Averaging processes in periodic media. Mathematical problems in mechanics of composite materials*, "Nauka", Moscow, 1984; English transl., Math. Appl. (Soviet Ser.), vol. 36, Kluwer Acad. Publ. Group, Dordrecht, 1989.

[BeLP]   Bensoussan A., Lions J.-L., Papanicolaou G., *Asymptotic analysis for periodic structures*, Stud. Math. Appl., vol. 5, North-Holland Publishing Co., Amsterdam-New York, 1978.

[BSu1]   Birman M. Sh., Suslina T. A., *Second order periodic differential operators. Threshold properties and homogenization*, Algebra i Analiz **15** (2003), no. 5, 1-108; English transl., St. Petersburg Math. J. **15** (2004), no. 5, 639–714.

[BSu2]   Birman M. Sh., Suslina T. A., *Threshold approximations with corrector for the resolvent of a factorized operator family*, Algebra i Analiz **17** (2005), no. 5, 69–90; English transl., St. Petersburg Math. J. **17** (2006), no. 5, 745–762.

[BSu3]   Birman M. Sh., Suslina T. A., *Homogenization with corrector term for periodic elliptic differential operators*, Algebra i Analiz **17** (2005), no. 6, 1–104; English transl., St. Petersburg Math. J. **17** (2006), no. 6, 897–973.

[BSu4]   Birman M. Sh., Suslina T. A., *Homogenization with corrector term for periodic differential operators. Approximation of solutions in the Sobolev class $H^1(\mathbb{R}^d)$*, Algebra i Analiz **18** (2006), no. 6, 1–130; English transl., St. Petersburg Math. J. **18** (2007), no. 6, 857–955.

[BSu5]   Birman M. Sh., Suslina T. A., *Operator error estimates in the homogenization problem for nonstationary periodic equations*, Algebra i Analiz **20** (2008), no. 6, 30–107; English transl., St. Petersburg Math. J. **20** (2009), no. 6, 873–928.

[DSu]   Dorodnyi M. A., Suslina T. A., *Homogenization of the hyperbolic equations with periodic coefficients*, in preparation.

[KamSu] Kamotski I., Suslina T., *Error estimates in homogenization of the nonstationary Schrödinger type equation*, in preparation.

[Ka]   Kato T., *Perturbation theory for linear operators*, Springer-Varlag, Berlin, 1995.

[Su1]   Suslina T. A., *On homogenization of periodic parabolic systems*, Funktsional. Analiz i ego Prilozhen. **38** (2004), no. 4, 86–90; English transl., Funct. Anal. Appl. **38** (2004), no. 4, 309–312.

[Su2]   Suslina T. A., *Homogenization of a periodic parabolic Cauchy problem*, Nonlinear Equations and spectral Theory, Amer. Math. Soc. Transl. Ser. 2, vol. 220, Amer. Math. Soc., Providence, RI, 2007, pp. 201–233.

[Su3]   Suslina T. A., *Homogenization of a periodic parabolic Cauchy problem in the Sobolev space $H^1(\mathbb{R}^d)$*, Math. Model. Nat. Phenom. **5** (2005), no. 4, 390–447.

[Sh1]   Shterenberg R. G., *An example of a periodic magnetic Schrödinger operator with degenerate lower edge of the spectrum*, Algebra i Analiz **16** (2004), no. 2, 177–185; English transl., St. Petersburg Math. J. **16** (2005), no. 2, 417–422.

[Sh2]   Shterenberg R. G., *On the structure of the lower edge of the spectrum of the periodic magnetic Schrödinger operator with small magnetic potential*, Algebra i Analiz **17** (2005), no. 5, 232–243; English transl., St. Petersburg Math. J. **17** (2006), no. 5, 865–873.

[Sh3]   Shterenberg R. G., *Homogenization of the periodic magnetic Schrödinger operator with degenerate lower edge of the spectrum*, Zapiski nauch. semin. POMI, v. 336, 2006, 264–273; English transl., Journal of Mathematical Sciences, v. 143, no. 2, 2007, 3004–3009.

[V]   Vasilevskaya E. S., *A periodic parabolic Cauchy problem: Homogenization with corrector*, Algebra i Analiz **21** (2009) no. 1, 3–60; English transl., St. Petersburg Math. J. **21** (2010), no. 1, 1–41.

[VSu]   Vasilevskaya E. S., Suslina T. A., *Homogenization of parabolic and elliptic periodic operators in $L_2(\mathbb{R}^d)$ with the first and second correctors taken into account*, Algebra i Analiz **24** (2012), no. 2, 1–103; English transl., St. Petersburg Math. J. **24** (2013), no. 2, 185–261.

[Zh]   Zhikov V. V., *On some estimates of homogenization theory*, Dokl. Ros. Akad. Nauk **406** (2006), no. 5, 597-601; English transl., Dokl. Math. **73** (2006), 96–99.

[ZhKO]   Zhikov V. V., Kozlov S. M., Olejnik O. A., *Homogenization of differential operators*, "Nauka", Moscow, 1993; English transl., Springer-Verlag, Berlin, 1994.

[ZhPas1] Zhikov V. V., Pastukhova S. E., *On operator estimates for some problems in homogenization theory*, Russ. J. Math. Phys. **12** (2005), no. 4, 515-524.

[ZhPas2] Zhikov V. V., Pastukhova S. E., *Estimates of homogenization for a parabolic equation with periodic coefficients*, Russ. J. Math. Phys. **13** (2006), no. 2, 224–237.

St. Petersburg State University, Department of Physics, Ul'yanovskaya 3, Petrodvorets, St. Petersburg, 198504, Russia

E-mail address: suslina@list.ru